Math 54 Spring 2005

Solutions to Homework Set 21

Section 5.2

24. *Show that if v is an eigenvector of A with eigenvalue λ, then v is also an eigenvector of A_k for any positive integer k.*

    $A^k v = A(A(\ldots(Av)\ldots))_{\text{(k A's)}} = A(A(\ldots(A(\lambda v))\ldots))_{\text{(k-1 A's)}} = A(A(\ldots(A(\lambda^2 v))\ldots))_{\text{(k-2 A's)}}$
    $= \ldots = \lambda^k v$. So v is an eigenvector with eigenvalue λ^k.

29. *A square matrix B is nilpotent if $B^k = 0$ for some integer $k > 1$. Show that 0 is the only eigenvalue of a nilpotent matrix.*

    Let λ be an eigenvalue of a nilpotent matrix B, and let k be a positive integer k such that $B^k = 0$. Then from Exercise 24 we see that λ^k is an eigenvalue of the zero matrix. I.e. there is a nonzero vector v s.t. $0v = \lambda^k v$. This implies $\lambda^k = 0$ which implies $\lambda = 0$.

30. *A square matrix C is idempotent if $C^2 = C$. What are the possible eigenvalues of an idempotent matrix?* Let

    λ be an eigenvalue of a idempotent matrix C and v be a corresponding eigenvector. Then $\lambda v = Cv = C^2 v = C(Cv)) = C(\lambda v) = \lambda^2 v \Rightarrow \lambda^2 = \lambda$, so $\lambda = 0 or 1$. Both of these values are possible. For example the identity matrix is idempotent and has eigenvalue 1. The zero matrix is idempotent and has eigenvalue 0.

31. *Suppose A is a 3x3 matrix with eigenvalues 0,2,4 and corresponding eigenvectors u_1, u_2, u_3.*

    a) *Find bases for NS(A) and CS(A) [Hint: $y \in CS(A) \Rightarrow y = Ax$]*

    b) *Solve $Ax = u_2 + u_3$.*

    c) *Show that $Ax = u_1$ has no solution.*

    Solution:

    a) Suppose v is in the nullspace of A. Then Av=0, so either v=0 or v is an eigenvector with eigenvalue 0, so it must be a nonzero multiple of u_1. We conclude that NS(A)=span(u_1). For CS(A), follow the hint. We $Au_2 = 2u_2, Au_3 = 4u_3$, so u_2 and u_3 are in CS(A). $dim(CS(A)) = 3 - dim(NS(A)) = 3 - 1 = 2$. Let's show u_2, u_3 are linearly independent. Suppose not, then $\exists a s.t. u_2 = au_3$. Then $2u_2 = Au_2 = Aau_3 = aAu_3 = a4u_3 = 4au_3 = 4u_2$, which is not possible since $u_2 \neq 0$. We have, $u_2, u_3 \in CS(A)$ and they are linearly independent, so they are a basis.

    b) $Ax = u_2 + u_3 = A(1/2u_2) + A(1/4u_3) \Leftrightarrow A(x - 1/2u_2 - 1/4u_3) = 0 \Leftrightarrow x - 1/2u_2 - 1/4u_3 \in NS(A) \Leftrightarrow \exists a s.t. x - 1/2u_2 - 1/4u_3 = au_1 (by a) \Leftrightarrow x = 1/2u_2 + 1/4u_3 + au_1$.

    c) $Ax = u_1 \Rightarrow u_1 \in CS(A) \Rightarrow \exists a, b s.t. u_1 = au_2 + bu_3 \Rightarrow 0Au_1 = A(au_2 + bu_3) = aAu_2 + bAu_3 = 2au_2 + 3bu_3$ but we showed above that $u_2 and u_3$ are linearly independent, so a and b must be zero but then u_1 must be zero, which is not the case.

34. *Find the eigenvalues and eigenvectors of* $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ *and of* $B = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

$det(\lambda I - B) = det(\begin{bmatrix} \lambda - a & -b \\ -b & \lambda + a \end{bmatrix}) = (\lambda - a)(\lambda + a) - bb = \lambda^2 - a^2 - b^2 =$

$= (\lambda - \sqrt{a^2 + b^2})(\lambda + \sqrt{a^2 + b^2})$, so the eigenvalues are $\pm\sqrt{a^2 + b^2}$.

To find the eigenvectors we need to find $NS(\begin{bmatrix} \sqrt{a^2 + b^2} - a & -b \\ -b & \sqrt{a^2 + b^2} + a \end{bmatrix})$ and

$NS(\begin{bmatrix} sqrta^2 + b^2 - a & -b \\ -b & -\sqrt{a^2 + b^2} + a \end{bmatrix})$.

Let's find the first nullspace. (assume b is not zero) Note that if we multiply the first row by $\frac{\sqrt{a^2+b^2}+a}{b}$ and add to the second row, we get the following matrix $\begin{bmatrix} \sqrt{a^2 + b^2} - a & -b \\ 0 & 0 \end{bmatrix}$. Now v=(x,y) is in $NS(\sqrt{a^2 + b^2}I - B) \Leftrightarrow (\sqrt{a^2 + b^2} - a)x + (-b)y = 0 \Leftrightarrow y = \frac{\sqrt{a^2+b^2}-a}{b}x$ so $NS(\sqrt{a^2 + b^2}I - B) = span([b, \sqrt{a^2 + b^2} - a)]^T)$. You can check this formula works in the case b=0 too.

Similarly $NS(-\sqrt{a^2 + b^2}I - B) = span([b, -\sqrt{a^2 + b^2} - a)]^T)$.

For A just plug in a=3 and b=4 in the above.