Math 54 Spring 2005

Solutions to Homework

Section 10.2

Problems 1-8: Determine whether the given function is periodic. If so, find the fundamental period.

1. $\sin 5x$.

   In general if $f(x)$ is periodic with fundamental period T then $f(ax), a > 0$ is periodic with fundamental period $\frac{T}{a}$. This is equivalent to saying $f(x) = f(x+T)\forall x$ if and only if $f(ax) = f(a(x+\frac{T}{a}))\forall x$, which is obvious.

   $\sin x$ is periodic with period $2\pi \Rightarrow \sin 5x$ is periodic with period $\frac{1}{5}2\pi = \frac{2\pi}{5}$.

2. $\cos 2\pi x$.

   $\cos x$ is periodic with period $2\pi \Rightarrow \cos 2\pi x$ is periodic with period $\frac{1}{2\pi}2\pi = 1$.

3. $\sinh 2x$.

   Suppose $\sinh 2x$ is periodic with period T. If $f(x)$ is differentiable and periodic with period T then its derivative is also periodic with the same period $[f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{f(x+T+h)-f(x+T)}{h} = f'(x+T)]$

   So we get that then $\cosh 2x$ should also be periodic with period T. But then their difference $e^{2x} = \cosh 2x - \sinh 2x$ is also periodic with period T. This means $\forall x e^{2x} = e^{2x+2T} \Rightarrow e^{2T} = 1 \Rightarrow 2T = 0 \Rightarrow T = 0$ contradiction, since T is the period so must be positive. We have a contradiction, $\Rightarrow \sinh 2x$ is not periodic.

Problems 13-18: Graph the function and find its Fourier series.

14. $f(x) = \left\{ \begin{array}{ll} 1, & -L \le x < 0 \\ 0, & 0 \le x < L \end{array} \right\}, f(x+2L) = f(x)$.

    $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{0} 1dx + \frac{1}{L}\int_{0}^{L} 0dx = \frac{1}{L}(x|_{-L}^{0}) = 1$

    $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{0} 1\cdot\cos\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} 0\cdot\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{0}\cos\frac{n\pi x}{L}dx = \frac{1}{L}(\frac{L}{n\pi}\sin\frac{n\pi x}{L}|_{-L}^{0}) = -\frac{1}{n\pi}\sin(-n\pi) = \frac{1}{n\pi}\sin(n\pi) = 0$

    $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{0} 1\cdot\sin\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} 0\cdot\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{0}\sin\frac{n\pi x}{L}dx = -\frac{1}{L}(\frac{L}{n\pi}\cos\frac{n\pi x}{L}|_{-L}^{0}) = -\frac{1}{n\pi}(1-\cos(-n\pi)) = \frac{1}{n\pi}(\cos(n\pi)-1)$

    The Fourier series of f is

    $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n\pi}(\cos(n\pi)-1)\sin\frac{n\pi x}{L} = \frac{1}{2}+\sum_{n=1,nodd}^{\infty}\frac{1}{n\pi}(-2)\sin\frac{n\pi x}{L} = \frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{L}$

15. $f(x) = \left\{ \begin{array}{ll} x, & -\pi \le x < 0 \\ 0, & 0 \le x < \pi \end{array} \right\}, f(x+2\pi) = f(x)$.

    $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} xdx + \frac{1}{\pi}\int_{0}^{\pi} 0dx = \frac{1}{\pi}(\frac{x^2}{2}|_{-\pi}^{0}) = -\frac{\pi}{2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cdot\cos nxdx + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{0} xd\left(\frac{\sin nx}{n}\right) = \frac{1}{\pi}x\left(\frac{\sin nx}{n}\right)|_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}\frac{\sin nx}{n}dx = -\frac{1}{\pi}\int_{-\pi}^{0}\frac{\sin nx}{n}dx = \frac{1}{\pi}\frac{\cos nx}{n^2}|_{-\pi}^{0} = \frac{1}{\pi}(\frac{1}{n^2}(1-\cos(-n\pi)) = \frac{1}{n^2\pi}(1-\cos(n\pi))$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{\pi}dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cdot\sin nxdx + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nxdx = -\frac{1}{\pi}\int_{-\pi}^{0} xd\left(\frac{\cos nx}{n}\right) = -\frac{1}{\pi}x\left(\frac{\cos nx}{n}\right)|_{-\pi}^{0} + \frac{1}{\pi}\int_{-\pi}^{0}\frac{\cos nx}{n}dx = -\frac{\cos(-n\pi)}{n} + \frac{1}{\pi}\frac{\sin nx}{n^2}|_{-\pi}^{0} = -\frac{\cos(n\pi)}{n} = \frac{(-1)^{n+1}}{n}$

The Fourier series of f is

$-\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}(1-\cos(n\pi))\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right) = -\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{2\cos((2n-1)x)}{(2n-1)^2\pi} + \frac{(-1)^{n+1}\sin(nx)}{n}\right)$

27. *Suppose that g is an integrable periodic function with period T.*

   a) *If $0 \le a \le T$, show that $\int_0^T g(x)dx = \int_a^{a+T} g(x)dx$*
   $I := \int_0^T g(x)dx = \int_0^a g(x)dx + \int_a^T g(x)dx$
   Make a change of variable $y = x + T$ for the first integral. Get
   $I = \int_T^{a+T} g(y-T)dy + \int_a^T g(x)dx = \int_T^{a+T} g(y)dy + \int_a^T g(x)dx$
   Now change the variable y to x.
   $I = \int_T^{a+T} g(x)dx + \int_a^T g(x)dx = \int_a^{a+T} g(x)dx$

   b) *Shot that for any value of a, not necessarily in $0 \le a \le T$, $\int_0^T g(x)dx = \int_a^{a+T} g(x)dx$*
   Let a be any number. Then since $T > 0$ there is an integer n such that $nT \le a < (n+1)T$ i.e. $0 \le a-nT < T$. Then part a) will give $I := \int_0^T g(x)dx = \int_{(a-nT)}^{(a-nT)+T} g(x)dx$. Now make change of variable $y = x + nT$. Get $I = \int_a^{a+T} g(y-nT)dy = \int_a^{a+T} g(y)dy = \int_a^{a+T} g(x)dx$

   c) *Shot that for any values of a and b, $\int_a^{a+T} g(x)dx = \int_b^{b+T} g(x)dx$*
   We know that for any a, $\int_0^T g(x)dx = \int_a^{a+T} g(x)dx$ and for any b, $\int_0^T g(x)dx = \int_b^{b+T} g(x)dx$. These together give $\int_a^{a+T} g(x)dx = \int_b^{b+T} g(x)dx$.

29. a) *Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a set of mutually orthogonal vectors in three dimensions, and let $\mathbf{u}$ be any three-dimensional vector. Show that $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$, where $a_i = \frac{\mathbf{u}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}$ (NOTE: even though it is not stated explicitly, but it is implied that $\mathbf{v}_i \neq 0$.*

   Let $\mathbf{w}$ be any vector in the given vector space. $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthogonal and nonzero $\Rightarrow$ they are linearly independent. Since we are in a three dimensional vector space, these vectors form a basis $\Rightarrow \exists b_1, b_2, b_3 : w = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3$. Now
   $A := (u - a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3, w) = (u, b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3) - (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3, b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3) = b_1(u,\mathbf{v}_1) + b_2(u,\mathbf{v}_2) + b_3(u,\mathbf{v}_3) - \sum_{i,j=1}^{3} a_i b_j(\mathbf{v}_i,\mathbf{v}_j)$
   By orthogonality if $i \neq j, (\mathbf{v}_i,\mathbf{v}_j) = 0$ so
   $A = b_1(u,\mathbf{v}_1) + b_2(u,\mathbf{v}_2) + b_3(u,\mathbf{v}_3) - a_1b_1(\mathbf{v}_1,\mathbf{v}_1) - a_2b_2(\mathbf{v}_2,\mathbf{v}_2) - a_3b_3(\mathbf{v}_3, v_3)$
   But $(u,\mathbf{v}_i) = a_i(\mathbf{v}_i,\mathbf{v}_i)$, so $A = 0$.
   So $(u - a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3, w) = 0 \forall w$. Take $w = u - a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. We get $||u - a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3|| = 0$, so $u - a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 0$.

   b) *Define the inner product (u,v) by*

   $$(u,v) = \int_{-L}^{L} u(x)v(x)dx.$$

   Also, let

$$\phi_n(x) = \cos(n\pi x/L), n = 0, 1, 2, ...;$$
$$\psi_n(x) = \sin(n\pi x/L), n = 1, 2, ...;$$

Show that Eq. (10)
$[\int_{-L}^{L} f(x)\cos(n\pi x/L)dx = a_0/2\int_{-L}^{L}\cos(n\pi x/L)dx + \sum_{m=1}\infty a_m \int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx$
$+\sum_{m=1}\infty b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx]$
can be written in the form

$$(f, \phi_n) = a_0/2 + \sum_{m=1}\infty a_m(\phi_m, \phi_n) + \sum_{m=1}\infty b_m(\psi_m, \phi_n) \text{ (v)}.$$

Obvious.

c) *Use Eq.(v) and the corresponding equation for (f, ψ_n), together with the orthogonality relations, to show that $a_n = \frac{(f,\phi)n)}{(\phi_n,\phi_n)}, n = 0, 1, 2, ...; b_n = \frac{(f,\psi_n)}{(\psi_n,\psi_n)}$.*
Since all of ϕ_n, ψ_m are pairwise orthogonal, we get that in (v) the only non-zero term on the right is $a_n(\phi_n, \phi_n)$, so we have $(f, \phi_n) = a_n(\phi_n, \phi_n)$ i.e. $a_n = \frac{(f,\phi)n)}{(\phi_n,\phi_n)}$. The equation for the b_n can be gotten in a similar way.