Solutions to Homework

Section 10.2

Problems 1-8: Determine whether the given function is periodic. If so, find the fundamental period.

1. $\sin 5x$.

In general if $f(x)$ is periodic with fundamental period $T$ then $f(ax), a > 0$ is periodic with fundamental period $\frac{T}{a}$. This is equivalent to saying $f(x) = f(x + T) \forall x$ if and only if $f(ax) = f(a(x + \frac{T}{a}))(\forall x)$, which is obvious.

$\sin x$ is periodic with period $2\pi \Rightarrow \sin 5x$ is periodic with period $\frac{1}{5}2\pi = \frac{2\pi}{5}$.

2. $\cos 2\pi x$.

$\cos x$ is periodic with period $2\pi \Rightarrow \cos 2\pi x$ is periodic with period $\frac{1}{2\pi}2\pi = 1$.

3. $\sinh 2x$.

Suppose $\sinh 2x$ is periodic with period $T$. If $f(x)$ is differentiable and periodic with period $T$ then its derivative is also periodic with the same period $[f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x + T)]$

So we get that then $\cosh 2x$ should also be periodic with period $T$. But then their difference $e^{2x} = \cosh 2x - \sinh 2x$ is also periodic with period $T$. This means $\forall x e^{2x} = e^{2x+2T} \Rightarrow e^{2T} = 1 \Rightarrow 2T = 0 \Rightarrow T = 0$ contradiction, since $T$ is the period so must be positive. We have a contradiction, $\Rightarrow \sinh 2x$ is not periodic.

Problems 13-18: Graph the function and find its Fourier series.

14. $f(x) = \begin{cases} 1, & -L \leq x < 0 \\ 0, & 0 \leq x < L \end{cases}$, $f(x + 2L) = f(x)$.

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx = \frac{1}{L} \int_{0}^{0} 1 dx + \frac{1}{L} \int_{0}^{L} 0 dx = \frac{1}{L} (\int_{-L}^{0} 1 dx) = 1$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{L} \int_{0}^{0} 1 \cos \frac{n\pi x}{L} dx + \frac{1}{L} \int_{0}^{L} 0 \cos \frac{n\pi x}{L} dx = \frac{1}{L} \int_{0}^{L} \cos \frac{n\pi x}{L} dx = \frac{1}{L} \left( \frac{L}{n\pi} \sin \frac{n\pi x}{L} \right) = -\frac{1}{n\pi} \sin (-n\pi) = \frac{1}{n\pi} \sin (n\pi) = 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{L} \int_{0}^{0} 1 \sin \frac{n\pi x}{L} dx + \frac{1}{L} \int_{0}^{L} 0 \sin \frac{n\pi x}{L} dx = \frac{1}{L} \int_{0}^{L} \sin \frac{n\pi x}{L} dx = -\frac{1}{L} \left( \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right) = -\frac{1}{n\pi} (1 - \cos (-n\pi)) = \frac{1}{n\pi} (\cos (n\pi) - 1)$$

The Fourier series of $f$ is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} (\cos (n\pi) - 1) \sin \frac{n\pi x}{L} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} (-2) \sin \frac{n\pi x}{L} = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{L}$$

15. $f(x) = \begin{cases} x, & -\pi \leq x < 0 \\ 0, & 0 \leq x < \pi \end{cases}$, $f(x + 2\pi) = f(x)$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{0}^{0} x dx + \frac{1}{\pi} \int_{0}^{\pi} 0 dx = \frac{1}{\pi} \left( \frac{x^2}{2} \right) \Big|_{0}^{\pi} = -\frac{\pi}{2}$$
The Fourier series of \( f \) is

\[
a_n = \frac{1}{
\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n x \pi}{\pi} \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \cos n x \, dx + \frac{1}{\pi} \int_{0}^{\pi} 0 \cdot \cos n x \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos n x \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos n x \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos n x \, dx
\]

Define the inner product

\[
\langle u, v \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) v(x) \, dx
\]

27. Suppose that \( g \) is an integrable periodic function with period \( T \).

a) If \( 0 \leq a \leq T \), show that \( \int_{0}^{T} g(x) \, dx = \int_{a}^{a+T} g(x) \, dx \)

b) \( a \) be any number. Then since \( T > 0 \therefore \pi \), there is an integer \( n \) such that \( n T < a < (n+1)T \).

29. a) Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) be a set of mutually orthogonal vectors in three dimensions, and let \( \mathbf{u} \) be any three-dimensional vector. Show that \( \mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 \), where \( a_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} \) (NOTE: even though it is not stated explicitly, but it is implied that \( \mathbf{v}_i \neq 0 \).

Let \( \mathbf{w} \) be any vector in the given vector space. \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are orthogonal and nonzero \( \Rightarrow \) they are linearly independent. Since we are in a three dimensional vector space, these vectors form a basis \( \Rightarrow \exists \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 : \mathbf{w} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + b_3 \mathbf{v}_3 \).

By orthogonality if \( i \neq j \), \( \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \) so

b) Define the inner product \( \langle u, v \rangle \) by

\[
\langle u, v \rangle = \int_{-L}^{L} u(x)v(x) \, dx.
\]

Also, let
\phi_n(x) = \cos(n\pi x/L), n = 0, 1, 2, ...;
\psi_n(x) = \sin(n\pi x/L), n = 1, 2, ...;

Show that Eq. (10)
\[
\int_{-L}^{L} f(x) \cos(n\pi x/L) \, dx = a_0/2 \int_{-L}^{L} \cos(n\pi x/L) \, dx + \sum_{m=1}^{\infty} \infty a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \, dx
\]
\[
+ \sum_{m=1}^{\infty} \infty b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \, dx
\]
can be written in the form
\[
(f, \phi_n) = a_0/2 + \sum_{m=1}^{\infty} \infty a_m (\phi_m, \phi_n) + \sum_{m=1}^{\infty} \infty b_m (\psi_m, \phi_n) \text{ (v).}
\]

Obvious.

(c) Use Eq.(v) and the corresponding equation for \((f, \psi_n)\), together with the orthogonality relations, to show that
\[
a_n = (f, \phi_n) \quad n = 0, 1, 2, ...; \quad b_n = (f, \psi_n) / (\psi_n, \psi_n).
\]

Since all of \phi_n, \psi_m are pairwise orthogonal, we get that in (v) the only non-zero term on the right is \(a_n(\phi_n, \phi_n)\), so we have \((f, \phi_n) = a_n(\phi_n, \phi_n)\) i.e. \(a_n = (f, \phi_n) / (\phi_n, \phi_n)\). The equation for the \(b_n\) can be gotten in a similar way.