Solutions to Homework

Section 10.1

Problems 1-13: Either solve the given boundary value problem or else show it has no solution.

1. \( y'' + y = 0, y(0) = 0, y'(\pi) = 1. \)
   \[ r^2 + 1 = 0, \text{ so } r = \pm i, \text{ so } y = c_1 \cos x + c_2 \sin x. \] This is the general solution. \( y' = -c_1 \sin x + c_2 \cos x. \) Plug in the initial values. We get \( c_1 = 0, c_2 = 1, \) so the solution is \( y = \sin x. \)

3. \( y'' + y = 0, y(0) = 0, y'(L) = 0. \)
   \[ r^2 + 1 = 0, \text{ so } r = \pm i, \text{ so } y = c_1 \cos x + c_2 \sin x. \] This is the general solution. Plug in the initial values. We get \( c_1 = 0, c_1 \cos L + c_2 \sin L = 0, \text{ i.e. } c_2 \sin L = 0. \) If \( L = n\pi \) for some integer \( n, \) then \( \sin L = 0 \) so \( c_2 \) can be anything and the solution is \( y = c \sin x, \) otherwise \( \sin L \neq 0, \text{ so } c_2 = 0 \) and the only solution is \( y = 0. \)

5. \( y'' + y = x, y(0) = 0, y'(\pi) = 0. \)
   First solve the homogeneous equation \( y'' + y = 0, r^2 + 1 = 0, \text{ so } r = \pm i, \text{ so } y = c_1 \cos x + c_2 \sin x. \) This is the general solution of the homogeneous equation. Now, we look for a particular solution of the initial equation of the form \( ax + b. \) Since \( (ax + b)'' = 0, \) plugging this in the equation gives \( ax + b = x, \text{ so } a = 1, b = 0. \) So the general solution to the original equation is \( y = c_1 \cos x + c_2 \sin x + x. \) Now plug in the initial values. We get \( c_1 = 0, -c_1 + \pi = 0, \) which is not possible so the boundary value problem has no solution.

7. \( y'' + 4y = \cos x, y(0) = 0, y(\pi) = 0. \)
   First solve the homogeneous equation \( y'' + 4y = 0, r^2 + 4 = 0, \text{ so } r = \pm 2i, \text{ so } y = c_1 \cos 2x + c_2 \sin 2x. \) This is the general solution of the homogeneous equation. Now, we look for a particular solution of the initial equation of the form \( a \cos x + b \sin x. \) Plugging this in the equation gives \( -a \cos x - a \sin x + 4a \cos x + 4b \sin x = \cos x, \text{ so } 3a = 1, 3b = 0, \text{ i.e. } a = 1/3, b = 0. \) So the general solution to the original equation is \( y = c_1 \cos 2x + c_2 \sin 2x + 1/3 \cos x. \) Now plug in the initial values. We get \( c_1 + 1/3 = 0, c_1 - 1/3 = 0, \) which is not possible so the boundary value problem has no solution.

8. \( y'' + 4y = \sin x, y(0) = 0, y(\pi) = 0. \)
   First solve the homogeneous equation \( y'' + 4y = 0, r^2 + 4 = 0, \text{ so } r = \pm 2i, \text{ so } y = c_1 \cos 2x + c_2 \sin 2x. \) This is the general solution of the homogeneous equation. Now, we look for a particular solution of the initial equation of the form \( a \cos x + b \sin x. \) Plugging this in the equation gives \( -a \cos x - a \sin x + 4a \cos x + 4b \sin x = \sin x, \text{ so } 3a = 0, 3b = 1, \text{ i.e. } a = 0, b = 1/3. \) So the general solution to the original equation is \( y = c_1 \cos 2x + c_2 \sin 2x + 1/3 \sin x. \) Now plug in the initial values. We get \( c_1 = 0, c_1 = 0, \) so the solution is \( y = c \sin 2x + 1/3 \sin x. \)

11. \( x^2y'' - 2xy' + 2y = 0, y(1) = -1, y(2) = 1. \)
    If you have a differential equation of the form \( ax^2y'' + bxy' + cy = 0, \) the substitution \( t = \ln x, \text{ i.e. } x = e^t \) is useful. When you do the substitution, you have to change everything to the t
variable. In particular the $y''$ means derivative with respect to $x$, and we have to change it to a derivative with respect to $t$. Here is how it can be done.

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} (\frac{d\ln x}{dx}) = \frac{dy}{dt} \left(\frac{1}{x}\right).$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx}\right) = \frac{d}{dt} \left(\frac{dy}{dx}\right) \frac{dt}{dx} = \frac{d}{dt} \left(\frac{dy}{dx}\right) \left(\frac{1}{x}\right) = \frac{d}{dt} \left(\frac{dy}{dx}\right) (1/x) = \left(\frac{dy}{dt}\right) \left(\frac{1}{x}\right).$$

When we plug these in the differential equation we get the following differential equation with respect to $t$: $(y'' - y') - 2y' + 2y = 0$ Let’s solve this. $r^2 - 3r + 2 = 0$ so $r = 1, 2$. $y = c_1 e^t + c_2 e^{2t}$. With switching back to $x$ we get $y = c_1 x + c_2 x^2$. Plugging in the initial values gives $c_1 + c_2 = -1, 2c_1 + 4c_2 = 1$ which gives $c_2 = 3/2, c_1 = -5/2$ which gives us $y = -5/2x + 3/2x^2$ as a solution of the boundary value problem.

13. $x^2 y'' + 5xy' + (4 + \pi^2)y = 0, y(1) = 0, y(\pi) = 0.$

If you have a differential equation of the form $ax^2 y'' + bxy' + cy = 0$, the substitution $t = \ln x, i.e. x = e^t$ is useful. (see 10.1.11 for details)

When we do the substitution we get the following differential equation with respect to $t$: $(y'' - y') - 2y' + 2y = 0$ First solve the homogeneous equation. $r^2 + 4r + 4 + \pi^2 = 0$ i.e. $(r + 2)^2 = -\pi^2$ so $r = -2 \pm \pi i \Rightarrow y = e^{-2t}(c_1 \cos \pi t + c_2 \sin \pi t)$. This gives the general solution of the homogeneous equation. Now, find a particular solution. Look for one of the form $at + b$.

Plugging thin in the equation gives $5a + (4 + \pi^2)(at + b) = t$ so $(4 + \pi^2)a = 1.5a + (4 + \pi^2)b = 0 \Rightarrow a = \frac{1}{(4 + \pi^2)^2}, b = \frac{5}{(4 + \pi^2)^2}$. The general solution to the differential equation then is $y = e^{-2t}(c_1 \cos \pi t + c_2 \sin \pi t) + \frac{1}{(4 + \pi^2)^2} t - \frac{5}{(4 + \pi^2)^2} \ln x$. By switching back to $x$ we get $y = \frac{1}{x^2}(c_1 \cos \pi \ln x + c_2 \sin \pi \ln x) + \frac{1}{(4 + \pi^2)^2} \ln x - \frac{5}{(4 + \pi^2)^2}$. Plugging in the initial values gives $c_1 - \frac{5}{(4 + \pi^2)^2} = 0, -\frac{1}{\pi} c_1 + \frac{1}{(4 + \pi^2)^2} = 0$ which is not possible, so the boundary value problem has no solution.

15. $y'' + \lambda y = 0, y'(0) = 0, y(\pi) = 0.$

$r^2 + \lambda = 0, so r = \pm \sqrt{-\lambda}$. Now consider two cases. First case $\lambda \leq 0$ i.e. $r$ is real and $y = c_1 e^{-\sqrt{-\lambda} x} + c_2 e^{\sqrt{-\lambda} x}$. Plug in the initial values. We get $\sqrt{-\lambda} c_1 - \sqrt{-\lambda} c_2 = 0, c_1 e^{-\sqrt{\lambda} \pi} + c_2 e^{\sqrt{-\lambda} \pi}$. Since the equations for $c_1, c_2$ are linearly independent, the only solution is $c_1 = c_2 = 0$ so $y = 0$ i.e. $y$ is not an eigenvector.

Second case: $\lambda > 0$, i.e. $r$ is purely imaginary. Then $r = \pm \sqrt{\lambda} i$ and the solution is $y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$. $y' = -\sqrt{\lambda} c_1 \sin \sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos \sqrt{\lambda} x$. Plugging in the initial values gives $\sqrt{\lambda} c_2 = 0, c_1 \cos \sqrt{\lambda} \pi + c_2 \sin \sqrt{\lambda} \pi \ i.e. c_2 = 0, c_1 \cos \sqrt{\lambda} \pi = 0$. Now, if $\sqrt{\lambda} \pi$ is not of the form $-\frac{\pi}{2} + n\pi$, then $c_1 \cos \sqrt{\lambda} \pi \neq 0$ so $c_1 = 0 \Rightarrow y = 0$ so $y$ is not an eigenvector. But if $\sqrt{\lambda} \pi$ is of the form $-\frac{\pi}{2} + n\pi$ then $\cos \sqrt{\lambda} \pi = 0$ so $c_1$ can be anything. This happens when $\sqrt{\lambda} = n - 1/2$ and $\lambda > 0$ i.e. $\lambda = (n - 1/2)^2$ for some positive integer $n$. These are the eigenvalues. The corresponding eigenfunctions are $y = c_1 \sin (n - 1/2)x$. 

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