Solutions for 1.4 and 3.1

February 4th

Section 1.4

4. *Find the inverse e^{-1} of the given elementary row operation e and find the matrices associated with e and e^{-1}. e is "Add 7 times the fourth row to the second row of a 4×8 matrix."*

The inverse of e is "subtract 7 times the fourth row from the second row." We find the associated matrix of a row operation on an $m \times n$ matrix by performing that row operation on an $m \times m$ identity matrix. So, denoting the associated matrix by the same name as the row operation,

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. *Find the elementary matrix, E, which adds 3 times the second row to the first row of the matrix $A = \begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$. Compute $e(A)$ and EA.*

$$E = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } e(A) = EA = \begin{bmatrix} 26 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}.$$

10. *Is the following an elementary matrix and, if so, to what row operation does it correspond?*

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is an elementary matrix corresponding to swapping the first and third rows.

12. *Is the following an elementary matrix and, if so, to what row operation does it correspond?*

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

is not an elementary matrix since it does not correspond to applying *one* elementary row operation - it does correspond to applying two.

14. *Is the following an elementary matrix and, if so, to what row operation does it correspond?*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is not an elementary matrix since no row operation on the identity could give all zeroes as the third column or row - we may not multiply a row by zero.

16. *Find elementary matrices E_3 and E_4 such that $E_3B = C$ and $E_4C = B$ and explain their relation.*
($B = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 \\ 8 & 9 & 10 & 11 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.)
Since we obtain C from B by swapping the second and third rows of B and we obtain B from C by swapping the second and third rows of C,

$$E_3 = E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Not only are these matrices the same but they are also mutually (and thus self-)inverse.

20. *Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$. Find elementary matrices E_1 and E_2 such that $E_2E_1A = I$, write A and A^{-1} as a product of elementary matrices.*

We row row reduce A to I in two steps: first, e_1, we add -2 times row one to row two; second, e_2, we scale row 2 by 1/3. The corresponding elementary matrices are $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. Thus, we have $E_2E_1A = I$, in which case $A^{-1} = E_2E_1$ and $A = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1}$.

30. *Find the inverse of A if it exists and check by multiplying out, where $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.*

Using the augmented matrix technique, we get the following sequence of matrices:

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{-1}{2} & \frac{-5}{2} & \frac{-3}{2} & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -5 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 0 & 3 & -5 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & -2 & 2 & 2 \\ 0 & 1 & 0 & 3 & -5 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 & -5 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Hence, the inverse is $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -5 & -2 \\ 0 & 1 & 0 \end{bmatrix}$.

46. *Show that if $ad - bc = 0$ then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has no inverse.*

    Assume for a contradiction that A has an inverse. We note

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} ad - bc \\ cd - cd \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

    and that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b \\ -a \end{bmatrix} = \begin{bmatrix} ab - ba \\ bc - da \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

    Now by theorem (1.50) these must both be trivial zeroes - i.e. $a = b = c = d = 0$. But then A is the zero matrix which cannot have an inverse which contradicts our initial assumption that A did have an inverse. Thus this assumption must have been false and thus A has no inverse.

48. *For the given conditions state whether (i) A must be invertible, (ii) A may or may not be invertible or (iii) A is not invertible*

    (a) A is 3×3 and $A \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

    By theorem (1.50) A cannot be invertible - it has a non-trivial zero.

    (b) $A = BC$ and both B and C are invertible.

    Then $AC^{-1}B^{-1} = C^{-1}B^{-1}A = I$ so $A^{-1} = C^{-1}B^{-1}$.

    (c) $A = B + C$ and B and C are both invertible.

    A may or may not be invertible. If $B = I$ then both B and $-B$ are invertible. However $B + B = 2I$ is invertible and $B + (-B) = 0$ is not.

    (d)

$$A \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = A \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

    Then

$$A \begin{bmatrix} -3 \\ -3 \\ 5 \end{bmatrix} = A(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}) = A \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - A \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So A has a nontrivial zero and again is not invertible.

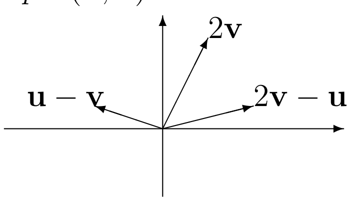
Section 3.1

1. $(-1, 1)$
2. $(4, -8, -7)$
3. $(-5, 11, -2)$
4. $(0, 0, 0)$
5. $(-1, 4, 6)$

10. $(144, -151, 169)$

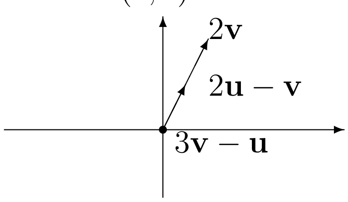
14. *Let* $\mathbf{v} = (1, 2)$

   (a) *Let* $\mathbf{u} = (-2, 3)$. *First sketch several linear combinations of* $\mathbf{v}$ *and* $\mathbf{u}$. *Then sketch Span*$(\mathbf{v}, \mathbf{u})$.

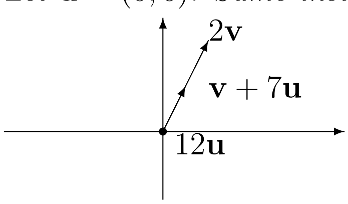
   The span is all of $\mathbb{R}^2$

   (d) *Let* $\mathbf{u} = (3, 6)$. *Same instructions as in (a).*

   The span is the line of slope 2 through $(0, 0)$

   (e) *Let* $\mathbf{u} = (0, 0)$. *Same instructions as in (a).*

   The span is the line of slope 2 through $(0, 0)$

16. *Look first at several cases in Exercises 14 and 15.*

   (a) *Let* $\mathbf{v} = (1, 2)$ *and suppose that* $\mathbf{u}$ *is any point in the plane. When is Span*$(\mathbf{v}, \mathbf{u})$ *a line (and describe that line)?*

   Notice that the span always includes any scalar multiple of $\mathbf{v}$, and hence it includes the line through $(0, 0)$ with slope 2. Thus, for the span to be a line we must have $\mathbf{u}$ on the given line, which is to say that $\mathbf{u}$ must be a scalar multiple of $\mathbf{v}$. If $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, the span is all of $\mathbb{R}^2$.

(b) *If* $\mathbf{v}$ *and* $\mathbf{u}$ *are any two points in the plane, describe the possibilities for Span*$(\mathbf{v}, \mathbf{u})$.

First, we could have both $\mathbf{u}$ and $\mathbf{v}$ equal to $(0, 0)$. In this case, the span consists only of the origin.

Otherwise, at least one of $\mathbf{u}$ and $\mathbf{v}$ is not $(0, 0)$, in which case the span contains the line through this point. If the other vector is a scalar multiple of this vector (including the case where the second vector is $(0, 0)$, the span is precisely this line.

Finally, we could have both $\mathbf{u}$ and $\mathbf{v}$ different from $(0, 0)$, and neither a scalar multiple of the other. In this case, the span is all of $\mathbb{R}^2$.

20a. The span is all of $\mathbb{R}^3$, as any (x, y, z) can be written as $(x, y, z) = (x - y)(1, 0, 0) + (y - z)(1, 1, 0) + z(1, 1, 1)$.

20b. The span is all of $\mathbb{R}^3$, as any (x, y, z) can be written as $(x, y, z) = (-x + z)(-1, 0, 0) + ((y + z)/2)(1, 1, 1) + ((z - y)/2)(1, -1, 1)$.

20c. The span is all of $\mathbb{R}^3$, as any (x, y, z) can be written as $(x, y, z) = (x + z - y)(1, 1, 1) + (y - z)(1, 1, 0) + (y - x)(0, 1, 1)$.

20d. The span is the same as the span of $(0, 1, 1)$ and $(1, -1, 0)$ (since $(1, 0, 1) = (0, 1, 1) + (1, -1, 0)$), which is clearly a plane.