Math 54W Spring 2005

Solutions to Homework Section 3.7

February 18th, 2005

2. *List the row vectors and the column vectors of the matrix* $\begin{pmatrix} 1 & 2 & 0 & -3 & 4 \\ 5 & 1 & -3 & 2 & -2 \end{pmatrix}$.

The row vectors are

$$(1,2,0,-3,4), \quad (5,1,-3,2,-2).$$

The column vectors are

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

5. *The matrix*

$$A = \begin{pmatrix} 2 & -4 & 3 & 1 \\ 0 & -3 & -2 & 7 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

is in row echelon form. Find a basis for its row space, find a basis for its column space, and determine its rank.

Since A is already in row echelon form, its nonzero rows form a basis for $RS(A)$ by Theorem 3.70. Since all of the rows are nonzero, a basis for $RS(A)$ is

$$(2,-4,3,1), \quad (0,-3,-2,7), \quad (0,0,4,1), \quad (0,0,0,5).$$

For the column space, we use Theorem 3.73, which says that the column vectors containing pivots form a basis for $CS(A)$. Since every column has a pivot, a basis for $CS(A)$ is

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -2 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 7 \\ 1 \\ 5 \end{pmatrix}.$$

8. We have $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$. This is row equivalent to $\mathbf{U} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. A basis for RS$\mathbf{U}$ consists of the vectors $(3,2,-1)$ and $(0,-1,7)$. Since RS$\mathbf{U}$ =RS$\mathbf{A}$, these also constitute a basis for RS$\mathbf{A}$. The first two columns of $\mathbf{U}$ constitute a basis for CS$\mathbf{U}$. Thus, the first two columns of $\mathbf{A}$, namely $(3,6,-3,0)$ and $(2,3,-1,-1)$, constitute a basis for CS$\mathbf{A}$. Since all the bases here contain two elements, we see rk$\mathbf{A} = 2$.

12. Note that $\mathbf{V} = \text{Span}\{(-2,4,1,4),(4,2,3,-1),(2,6,4,1)\} = \text{RS}\mathbf{A}$, where $\mathbf{A} = \begin{bmatrix} -2 & 4 & 1 & 2 \\ 4 & 2 & 3 & -1 \\ 2 & 6 & 4 & 1 \end{bmatrix}$. $\mathbf{A}$ is row equivalent to $\mathbf{U} = \begin{bmatrix} -2 & 4 & 1 & 2 \\ 0 & 10 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus a basis for $\mathbf{V} =$ RS$\mathbf{A}$ consists of the vectors $(-2,4,1,2)$ and $(0,10,5,3)$.

18. $\mathbf{A} = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$ is row equivalent to $\mathbf{U} = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -15 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus NS$\mathbf{A} = \{(\frac{2}{7}(15t+s) - 4t - s, \frac{1}{7}(15t+s), t, s) \mid t, s \in \mathbf{R}\}$, with basis
$\mathbf{B} = \{(\frac{2}{7}, \frac{15}{7}, 1, 0), (\frac{-5}{7}, \frac{1}{7}, 0, 1)\}$. This shows dimNS$\mathbf{A} = 2$. Since $\mathbf{U}$ has two pivots, we see rk$\mathbf{A} = 2$. Sure enough $2 + 2 = 4 = n$ in this case.

In exercises 22-24, determine if **b** *lies in the column space of A. If it does, express* **b** *as a linear combination of the columns of A.*

22. $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$.
The second column of A is $\frac{3}{2}$ times the first, so

$$CS(A) = \text{Span}\{\begin{pmatrix} 2 \\ -4 \end{pmatrix}\} = \{\begin{pmatrix} 2x \\ -4x \end{pmatrix} \mid x \in \mathbb{R}\}.$$

Since $\mathbf{b}$ cannot be expressed in the form $\begin{pmatrix} 2x \\ -4x \end{pmatrix}$, it does not lie in the column space.

24. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$.
The vector **b** lies in the column space if and only if **b** can be written as a linear combination

$$\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

for scalars a, b and c. So we have to solve the system of equations

$$\begin{array}{rcrcrcr} a & + & b & - & c & = & 2 \\ a & + & 2b & + & 2c & = & -3 \\ 2a & + & 3b & + & c & = & -1. \end{array}$$

A bit of row reduction

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

tells us nope, the system is inconsistent, so **b** is not in the column space.

Shortcut: For future reference, notice that the matrix associated to the system was just A itself, augmented by the vector **b**. So if you want to save time, skip the first two steps and jump right into the row reduction.

39. *Let*

$$A = \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find bases for RS, NS, CS and LNS. Find the rank of A and verify that $\dim RS + \dim NS = n$, $\dim CS + \dim LNS = m$.

Since A is already in row echelon form, a basis for the row space is given by the nonzero rows of A:

$$(1, 3, -2, 4), \quad (0, 0, 5, 1).$$

Since the row space has two basis vectors, A has rank 2.

For the null space, set up a system of equations and write everything in terms of the free variables:

$$NS(A) = \{\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid A\mathbf{x} = \mathbf{0}\} = \{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid 5z + w = 0,\ x + 3y - 2z + 4w = 0\}. = \{\begin{pmatrix} -3y - 22w/5 \\ y \\ -w/5 \\ w \end{pmatrix}$$

This tells us that

$$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -22/5 \\ 0 \\ -1/5 \\ 1 \end{pmatrix}$$

is a basis for $NS(A)$, and we can now verify that $\dim RS + \dim NS = 2 + 2 = 4$.

A basis for the column space consists of the columns of A which have pivots:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad , \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}.$$

Finally, a row vector $\mathbf{x} = (x, y, z)$ lies in the left null space $LNS(A)$ if and only if

$$(x, y, z) \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (0, 0, 0, 0),$$

and this happens if and only if

$$x = 0, \quad 3x = 0, \quad -2x + 5y = 0, \quad 4x + y = 0.$$

In the solution to this system, z is a free variable and $x = y = 0$. Thus the left null space consists of all vectors of the form $(0, 0, z)$. This is a one-dimensional space with basis $(0, 0, 1)$. We can now verify $\dim CS + \dim LNS = 2 + 1 = 3$.

43. *True or false?*

[(a)] *If A is an $n \times n$ matrix, then the row space of A is equal to the column space of A.* False. The 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

has row space spanned by $(1, 1)$ and column space spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. These are not the same.

[(b)] *Even if A is square, the column space of A can never equal the null space of A.*
False. The 2×2 matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

has $CS(A) = NS(A) = \{ \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R} \}$.

[(c)]*If A is an $m \times n$ matrix and the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{b}$ may or may not have a solution. But if it has a solution, that solution is unique.*
False. Let $m = n = 1$, A=(0) (the 1×1 zero matrix), $\mathbf{b} = (0)$. The equation $0x = 0$ has many solutions.

[(d)]*A 3×4 matrix never has linearly independent columns.*
True. Four vectors in $\mathbb{R}^3$ can never be linearly independent.

[(e)]*A 4×3 matrix must have linearly independent columns.*
False. The zero matrix doesn't have linearly independent columns. As you can see, the zero matrix is very useful for producing counterexamples!

44. We consider $\mathbf{A}$ as a collection of n columns in $\mathbf{R}^m$.

(a.) If these vectors are linearly independent, then they form a basis of CS$\mathbf{A}$, in which case rk$\mathbf{A}$ = dimCS$\mathbf{A}$ = n. We must have $n \leq m$ since you cannot have more than m linearly independent vectors in $\mathbf{R}^m$.

(b.) If these vectors span $\mathbf{R}^m$, we have by definition CS$\mathbf{A} = \mathbf{R}^m$, and hence rk$\mathbf{A}$ = dimCS$\mathbf{A}$ = dim$\mathbf{R}^m = m$. In this case, $n \geq m$ since one cannot have fewer than m vectors spanning $\mathbf{R}^m$.

(c.) If these vectors form a basis of $\mathbf{R}^m$, then both (a.) and (b.) hold, in which case $n = m =$ rk$\mathbf{A}$, and we see that $\mathbf{A}$ is a square, invertible matrix.

In Ex. 48 we suppose $\mathbf{A}$ *is an nxn matrix and has a right inverse* $\mathbf{B}$ *such that* $\mathbf{AB} = \mathbf{I}$.

[WARNING: We do not assume that $\mathbf{A}$ is invertible, as we are not told whether $\mathbf{BA} = \mathbf{I}$. In fact, this is exactly what we set out to prove!]

48. (a.) To show that CS$\mathbf{A} = \mathbf{R}^n$, it is enough to show that given any $\mathbf{v} \in \mathbf{R}^n$, we can find an $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{v}$. (c.f. Ex. 33) But notice that $\mathbf{v} = \mathbf{Iv} = \mathbf{ABv} = \mathbf{A}(\mathbf{Bv})$. Thus, setting $\mathbf{x} = \mathbf{Bv}$, we see that $\mathbf{Ax} = \mathbf{v}$, and we are done.

(b.) Since rk$\mathbf{A}$ = dimCS$\mathbf{A}$ = dim$\mathbf{R}^n = n$, we see $\mathbf{A}$ is invertible by 3.83.d.

(c.) Since $\mathbf{A}$ is invertible, there exists a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$. Take the equation $\mathbf{AB} = \mathbf{I}$. Multiplying both sides on the left by $\mathbf{A}^{-1}$, we get $\mathbf{B} = \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}$, proving the claim.