Math 54 Spring 2005

Solutions to Homework Section 3.6 (continued)

February 16th, 2005

23. *Find a basis for M_{mn}. What is* $\dim M_{mn}$*?*

For numbers i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$, let E_{ij} be the matrix whose i, j entry is 1 and all other entries are 0. The matrices E_{ij} span M_{mn} because any $m \times n$ matrix $A = (a_{ij})$ can be written as a linear combination:

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}.$$

It's also easy to see that the E_{ij}'s are linearly independent: the right-hand side of the above equation represents a linear combination of the E_{ij}'s. If this combination is the zero matrix, that means A is the zero matrix, so all the a_{ij}'s are zero.

So the E_{ij}'s form a basis for M_{mn}. Since there are m possible choices for i and n possible choices for j, there are mn basis vectors in total, so $\dim M_{mn} = mn$.

26. *Let W be the subspace of $C[0,1]$ spanned by $S = \{\sin^2 x, \cos^2 x, \cos 2x\}$.*

[a.]*Explain why S is not a basis for W.*

Since $\cos 2x = \cos^2 x - \sin^2 x$, S is linearly dependent, and thus not a basis for W.

[b.]*Find a basis for W.*

Let $\mathbf{B} = \{sin^2 x, cos^2 x\}$. If $c_0 sin^2 x + c_1 cos^2 x = 0$ (the zero function in $C[0,1]$), then evaluating the equation above at $x = 0$, we find that $c_1 = 0$, leaving $c_0 sin^2 x = 0$. Now, $\frac{\pi}{6} \in [0,1]$, so evaluating at $x = \frac{\pi}{6}$, we see that $\frac{c_0}{4} = 0$, from which we conclude that $c_0 = 0$. Thus the only way to have $c_0 sin^2 x + c_1 cos^2 x = 0$ is if $c_0 = c_1 = 0$. We conclude that $\mathbf{B}$ is linearly independent. Note that, $cos 2x \in Span(V)$ (by a.), and of course, $sin^2 x, cos^2 x \in V \subseteq Span(V)$. Thus S is contained in $Span(\mathbf{B})$, which is a subspace of W, hence $Span(S) \subseteq Span(\mathbf{B})$, by Theorem 3.40(b). But now, $W = Span(S)$, so V spans all of W. Therefore, $\mathbf{B}$ is a linearly independent set which spans W, so $\mathbf{B}$ is a basis for W.

[c.]*What is dim W?*

The above basis for W has 2 elements, so dim $W = 2$.

28. *Find the dimension of the nullspace of A.* $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$ *This is row equivalent to*

$\boldsymbol{U} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -4 & 0 & -1 \end{bmatrix}$

Thus, $NS\mathbf{A} = \{(\frac{-3}{8}s + \frac{t}{2}, -\frac{s}{4}, t, s) | t, s, \in \mathbf{R}\}$. A basis for $NS\mathbf{A}$ is $\{(\frac{-3}{8}, -\frac{1}{4}, 0, 1), (\frac{1}{2}, 0, 1, 0)\}$. Since this basis has two elements, $\dim NS\mathbf{A} = 2$.

36. Any 3×3 skew-symmetric matrix A has the form

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (1)$$

Thus the matrices

$$M_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

span the space of skew-symmetric 3×3 matrices. To see that M_1, M_2 and M_3 are independent, notice that $aM_1 + bM_2 = cM_3$ equals the matrix on the left side of (1), which cannot be the zero matrix unless $a = b = c = 0$. So M_1, M_2, M_3 is a basis.

44. *Contract the columns of* $A = \begin{pmatrix} 0 & 2 & 3 & -6 \\ 0 & 0 & -3 & 6 \end{pmatrix}$ *to a basis of* $\mathbb{R}^2$*, and expand the rows of* A *to a basis of* $\mathbb{R}^4$*.*

The second and third columns, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$, form a basis of $\mathbb{R}^2$.

To expand the rows to a basis of $\mathbb{R}^4$, we would like to add two of the standard basis vectors ϵ_1, ϵ_2, ϵ_3, ϵ_4. Notice that the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is in row echelon form with a pivot in every column, so its rows are linearly independent. Using Theorem 3.64(d), we conclude that the rows of A together with ϵ_1 and ϵ_4 form a basis for $\mathbb{R}^4$.