Math 54 Spring 2005

Solutions to Homework, Sections 3.5-3.6

February 14th

Section 3.5

In exercises 1-4, show by inspection that the vectors are linearly dependent.

1. $u_1 = (2, -1)$, $u_2 = (6, -3)$, *in* $\mathbb{R}^2$.
   Since $u_2 = 3u_1$ the vectors are linearly dependent.

2. $v_1 = (4, -1, 3)$, $v_2 = (2, 3, -1)$, $v_3 = (-1, 2, -1)$, $v_4 = (5, 2, 3)$, *in* $\mathbb{R}^3$.
   By Theorem 3.46, any set of k vectors in $\mathbb{R}^n$, with $k > n$, is linearly dependent. Here $n = 3$, $k = 4$.

3. $w_1 = (2, -1, 4)$, $w_2 = (5, 2, 3)$, $w_3 = (0, 0, 0)$, *in* $\mathbb{R}^3$.
   Any set of vectors which includes the zero vector is linearly dependent. In this case $0w_1 + 0w_2 + w_3 = 0$ is a nontrivial linear combination equaling zero.

4. $p_1 = -x + 3x^3$, $p_2 = 2x - 6x^3$, *in* P_3.
   Since $p_2 = -2p_1$ the polynomials are linearly dependent.

In exercise 8, determine if the given vectors span a line, or a plane, or something larger, and relate this to the fact that they are linearly dependent or independent.

8. $v_1 = (1, 2)$, $v_2 = (2, 1)$, *in* $\mathbb{R}^2$.
   Since v_2 is not a scalar multiple of v_1, these vectors are linearly independent and span all of $\mathbb{R}^2$.

In exercises 16-22, determine if the given vectors are linearly dependent or independent. Do this in an easy way, if possible.

16. $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(0, 0, 1, 1)$, $(1, 0, 0, 1)$, *in* $\mathbb{R}^4$.
    These vectors are linearly dependent if and only if the equation

$$a(1, 1, 0, 0) + b(0, 1, 1, 0) + c(0, 0, 1, 1) + d(1, 0, 0, 1) = (0, 0, 0, 0)$$

has a nontrivial solution (a, b, c, d), i.e. a solution in which a, b, c and d are not all zero. This equation is equivalent to the system

$$\begin{array}{ccccccccc} a & & & & & + & d & = & 0 \\ a & + & b & & & & & = & 0 \\ & & b & + & c & & & = & 0 \\ & & & & c & + & d & = & 0. \end{array}$$

Row-reduce the associated matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There is no pivot in the last column, so d is a free variable and the system has infinitely many solutions. Therefore the vectors are linearly dependent.

18. $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$, *in* $\mathbb{R}^4$.
To minimize the amount of work we need to do, notice that the solution to problem 16 amounted to row-reducing the matrix whose *columns* are the vectors in question, and checking whether every column has a pivot. For the current problem, the relevant matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lucky for us, this matrix is already in row-echelon form. Since there's a pivot in every column, there are no free variables in the corresponding system, and hence the vectors are linearly independent.

20. $p_1(x) = 1 + x + x^2$, $p_2(x) = 2 - x + 3x^2$, $p_3(x) = -1 + 5x - 3x^2$, *in* P_2.
Row-reduce the corresponding matrix:

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 5 \\ 1 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix}.$$

There's no pivot in the last column, so the polynomials are linearly dependent.

28. *If* $\{u_1, \ldots, u_n\}$ *is linearly independent, show that any nonempty subset is linearly independent.*
By rearranging the order of the vectors u_i, we may assume that our subset has the form $\{u_1, \ldots, u_k\}$ for some number k between 1 and n. To prove that u_1, …, u_k are linearly independent, we first assume that c_1, …, c_k are real numbers such that

$$c_1u_1 + \ldots + c_ku_k = 0.$$

We must prove that $c_1 = \ldots = c_k = 0$. The trick is to write the above equation in the form

$$c_1u_1 + \ldots + c_ku_k + 0u_{k+1} + \ldots + 0u_n = 0. \tag{1}$$

Now because u_1, …, u_n are linearly independent, all the coeficients in equation (1) must be zeros. It follows that c_1, …, c_k are all zero, which is what we wanted to show.

32. *Suppose that the vectors* u_1, u_2 *and* u_3 *are linearly dependent. Are the vectors* $v_1 = u_1 + u_2$, $v_2 = u_1 + u_3$, *and* $v_3 = u_2 + u_3$ *also linearly dependent?*
If a_1, a_2 and a_3 are scalars such that

$$a_1v_1 + a_2v_2 + a_3v_3 = 0, \tag{2}$$

writing the v's in terms of the u's, we get

$$a_1(u_1+u_2)+a_2(u_1+u_3)+a_3(u_2+u_3)=0$$

so

$$(a_1+a_2)u_1+(a_1+a_3)u_2+(a_2+a_3)u_3=0. \tag{3}$$

We are given that u_1, u_2 and u_3 are linearly dependent. This means there are scalars b_1, b_2 and b_3, not all equal to zero, such that

$$b_1u_1+b_2u_2+b_3u_3=0. \tag{4}$$

We'd like to choose the numbers a_1, a_2 and a_3 to make equation (3) true. From equation (4), we can always do this if it's possible to solve the system of equations

$$\begin{array}{ccccccc} a_1 & + & a_2 & & & = & b_1 \\ a_1 & & & + & a_3 & = & b_2 \\ & & a_2 & + & a_3 & = & b_3. \end{array} \tag{5}$$

A bit of row reduction

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

shows that yes! this system always has a solution. Now it is easy to check that this solution satisfies the original equation (2). To conclude that v_1, v_2 and v_3 are linearly dependent, we just have to check one more thing: that a_1, a_2 and a_3 are not all equal to zero. But this is easy: if the a's were all equal to zero, then a glance back at the system (5) shows that the b's would be too. But by their very definition, the b's are not all equal to zero, so that takes care of that.

Section 3.6

In excercises 1-6, explain why the given vectors do not form a basis for the given vector space.

1. $u_1=(1,-2)$, $u_2=(2,-1)$, $u_3=(3,5)$, *for* $\mathbb{R}^2$.
   A basis for $\mathbb{R}^2$ must have two elements.

2. $\mathbf{v}_1=(3,-1,2)$, $\mathbf{v}_2=(0,2,5)$.
   A basis for $\mathbb{R}^3$ must have three elements.

5. $q_1=1+2x+3x^2$, $q_2=2-x-x^2$, *for* P_2.
   Since $1,x,x^2$ forms a basis for P_2, all bases of P_2 must have three elements by Theorem 3.62.

6. $M_1=\begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 2 \end{bmatrix}$ $M_2=\begin{bmatrix} -2 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$ $M_3=\begin{bmatrix} 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix}$ $M_4=\begin{bmatrix} -1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ $M_5=\begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.
   By Example 11 in Section 3.6, dim M_{23}=6, so every basis for M_{23} has 6 elements, by Theorem 3.62, so the above five element set cannot be a basis for M_{23}.

In exercises 10-16, determine whether the given vectors form a basis for the given vector space.

10. $\mathbf{v}_1 = (2, 0)$ *and* $\mathbf{v}_2 = (3, 3)$.

    Since they are linearly independent and are two in number, they form a basis of $\mathbb{R}^2$.

16. $v_1 = (1, 1, 0, 0), v_2 = (0, 1, 1, 0), v_3 = (0, 0, 1, 1), v_4 = (1, 0, 0, 1)$ *for* $\mathbb{R}^4$

    Observe that $v_1 - v_2 + v_3 - v_4 = (0, 0, 0, 0)$, so $\{v_1, v_2, v_3, v_4\}$ is not linearly independent, and hence not a basis (Definition 3.50).