Math 54, Spring 2005

10.5 and 10.6 Homework Solutions

May 9th

Section 10.5

In Problems 1 and 3, determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

1. $xu_{xx} + u_t = 0$

   This is separable. We try $u(x,t) = X(x)T(t)$ and get $u_{xx} = X''(x)T(t)$ and $u_t = X(x)T'(t)$. Substituting into the equation we get $xX''T = XT'$, so $\frac{xX''}{X} = \frac{T'}{T}$. Since now both sides must be constant, we set $\frac{xX''}{X} = -\lambda$ and $\frac{T'}{T} = -\lambda$. From these we get the ordinary differential equations:

$$\begin{aligned} xX'' + \lambda X &= 0 \\ T' + \lambda T &= 0 \end{aligned}$$

3. $u_{xx} + u_{xt} + u_t = 0$

   This is separable. We try $u(x,t) = X(x)T(t)$ and substitute into the equation we get $X''T + X'T' + XT' = 0$, which implies $\frac{X''}{X'+X} = -\frac{T'}{T}$. Since now both sides must be constant, we set $\frac{X''}{X'+X} = -\lambda$ and $-\frac{T'}{T} = -\lambda$. From these we get the ordinary differential equations:

$$\begin{aligned} X'' + \lambda X + \lambda X &= 0 \\ T' - \lambda T &= 0 \end{aligned}$$

7. *Find the solution of the heat conduction problem*

$$\begin{aligned} 100u_{xx} = u_t, \qquad & 0 < x < 1, \qquad & t > 0; \\ u(0,t) = 0, \qquad & u(1,t) = 0, \qquad & t > 0; \\ & u(x,0) = \sin 2\pi x - \sin 5\pi x, \quad & 0 \le x \le 1; \end{aligned}$$

   The general solution to the heat equation with boundary conditions of this form is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin \frac{n\pi x}{L}.$$

   where the c_n are determined uniquely by

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$$

   In this case $L = 1$ and $\alpha^2 = 100$. $u(x,0)$ is already in the form of a sine series, where $c_2 = 1$, $c_5 = -1$, and the other c_n are 0. Therefore

$$u(x,t) = c_2 e^{-2^2\pi^2 100t/1^2} \sin \frac{2\pi x}{1} + c_5 e^{-5^2\pi^2 100t/1^2} \sin \frac{5\pi x}{1} = e^{-400\pi^2 t} \sin(2\pi x) - e^{-2500\pi^2 t} \sin(5\pi x).$$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at $0°C$ for all $t > 0$. In problems 9 and 10 find an expression for the temperature $u(x,t)$ if the initial temperature distribution in the rod is the given function. Suppose that $\alpha^2 = 1$.

9. $u(x,0) = 50, \quad 0 < x < 40.$

The sine series for the function $u(x,0)$ is computed as follows:

$$c_n = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}dx = -\frac{100}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}|_0^L = -\frac{100}{n\pi}(\cos(n\pi)-1) = \begin{cases} \frac{200}{n\pi} & n\text{odd} \\ 0 & n \text{ even} \end{cases}$$

Plugging this into the general solution with $L = 40$ yields

$$u(x,t) = \sum_{n \text{ odd}} \frac{200}{n\pi}\, e^{-n^2\pi^2 t/40^2}\sin\frac{n\pi x}{40}.$$

10.

$$u(x,0) = \begin{cases} x & 0 \le x \le 20, \\ 40 - x & 20 \le x \le 40 \end{cases}$$

The sine series for the function $u(x,0)$ is computed as follows:

$$c_n = \frac{2}{L}\int_0^{L/2} x\sin\frac{n\pi x}{L}dx + \frac{2}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx =$$

$$\frac{2}{L}(\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L})|_0^{L/2} + \frac{2}{L}(\frac{L(x-L)}{n\pi}\cos\frac{n\pi x}{L} - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L})|_{L/2}^{L} =$$

$$\frac{2}{L}(\frac{L^2}{n^2\pi^2}\sin\frac{n\pi L/2}{L}) + \frac{2}{L}(- - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi L/2}{L}).$$

For the last equality we used $\cos\frac{n\pi L/2}{L} = 0$ and $\sin\frac{n\pi 0}{L} = \sin\frac{n\pi L}{L} = 0$. Continuing the simplification, we have

$$c_n = \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Plugging this into the general solution with $L = 40$ yields

$$u(x,t) = \sum_{n=1}^{\infty}\frac{160}{n^2\pi^2}\sin\frac{n\pi}{2}\, e^{-n^2\pi^2 t/40^2}\sin\frac{n\pi x}{40}.$$

14. *For the rod in Problem 9:*

   d. *How long does it take for the entire rod to cool off to a temperature of no more than* $1^\circ C$
   Because of the symmetry of the initial temperature distribution and the boundary conditions, the warmest point is always at $x = 20$. We want to find the time t such that $u(20,t) = 1^\circ$C.

$$u(20,t) = \sum_{n \text{ odd}}\frac{200}{n\pi}\, e^{-n^2\pi^2 t/40^2}\sin\frac{n\pi 20}{40} \cong \frac{200}{\pi}\, e^{-\pi^2 t/40^2}$$

   Setting this to 1 and solving, we obtain $\ln\frac{\pi}{200} = -\pi^2 t/40^2$ implies $t \cong 673$. This is in seconds because α^2 has units cm^2/sec and L has units cm. This approximation is reasonable because in this time range the exponential term is about e^{-4} for $n = 1$ and e^{-36} for $n = 3$.

15. *For the rod in Problem 10:*

d. *How long does it take for the entire rod to cool off to a temperature of no more than 1° C*

Because of the symmetry of the initial temperature distribution and the boundary conditions, the warmest point is always at $x = 20$. We want to find the time t such that $u(20,t) = 1°\text{C}$.

$$u(20,t) = \sum_{n=1}^{\infty} \frac{160}{n^2\pi^2} \sin\frac{n\pi}{2}\ e^{-n^2\pi^2 t/40^2} \sin\frac{n\pi 20}{40} \cong \frac{160}{\pi^2}(\sin\frac{\pi}{2})\ e^{-\pi^2 t/40^2} \sin\frac{\pi 20}{40} = \frac{160}{\pi^2}\ e^{-\pi^2 t/40^2}$$

Setting this to 1 and solving, we obtain $\ln\frac{\pi^2}{160} = -\pi^2 t/40^2$ implies $t \cong 452$. This approximation is reasonable because in this time range the exponential term is about e^{-3} for $n = 1$ and e^{-12} for $n = 2$.

19a. *A silver rod 20 cm long is heated to a uniform temperature of 100° C. At $t = 0$ the ends of the bar are kept at 0° C. Find an expression for the temperature at any point at any time $t > 0$. Find how long will it take for the center to cool to 5° C.*

This is the same as Problem 9 except with different constants L, α, and initial temperature. For silver, $\alpha^2 = 1.71$.

$$u(x,t) = \sum_{n \text{ odd}} \frac{400}{n\pi}\ e^{-n^2 1.71\pi^2 t/20^2} \sin\frac{n\pi x}{20}.$$

$$u(10,t) = \sum_{n \text{ odd}} \frac{400}{n\pi}\ e^{-n^2 1.71\pi^2 t/20^2} \sin\frac{n\pi 10}{20} \cong \frac{400}{\pi}\ e^{-1.71\pi^2 t/20^2}$$

Setting this to 5 and solving, we obtain $\ln\frac{\pi}{80} = -1.71\pi^2 t/20^2$ implies $t \cong 77$. This approximation is reasonable because in this time range the exponential term is about e^{-3} for $n = 1$ and e^{-12} for $n = 2$.

Section 10.6

In Problems 1 and 4 find the steady-state solution of the heat equation $\alpha^2 u_{xx} = u_t$ that satisfies the given set of boundary conditions.

1. $u(0,t) = 10, \quad u(50,t) = 40$

The steady-state solution $v(x)$ must satisfy the differential equation with $v_t = 0$, which implies $v_{xx} = 0$. Therefore $v(x) = c_1 + c_2 x$, for some constants c_1 and c_2. $v(x)$ must also satisfy the boundary conditions $v(0) = 10$ and $v(50) = 40$ so $10 = v(0) = c_1$ and $40 = v(50) = c_1 + 50c_2 = 10 + 50c_2$. Thus $c_2 = \frac{3}{5}$ and $v(x) = 10 + \frac{3}{5}x$.

4. $u_x(0,t) = 0, \quad u(L,t) = T$

The steady-state solution $v(x)$ must satisfy the differential equation with $v_t = 0$, which implies $v_{xx} = 0$. Therefore $v(x) = c_1 + c_2 x$, for some constants c_1 and c_2. $v(x)$ must also satisfy the boundary conditions $v_x(0) = 0$ and $v(L) = T$ so $0 = v_x(0) = c_2$ and $T = v(L) = c_1 + Lc_2 = c_1$. Thus $v(x) = T$.

9 *Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25° C. Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0° C while the end $x = 20$ is heated to 60° C, and both are thereafter maintained at those temperatures.*

a. *Find the temperature distribution in the rod at any time.*

In this problem $L = 20$, $T_1 = 0$, $T_2 = 60$, and $\alpha^2 = .86$. The steady-state solution is

$$v(x) = (T_2 - T_1)\frac{x}{L} + T_1 = 3x.$$

We will express $u(x,t)$ as the sum of the steady-state solution and another solution $w(x,t)$ that depends on time: $u(x,t) = v(x) + w(x,t)$. As shown in the discussion in 10.6 (p. 614), $w(x,t)$ also satisfies $\alpha^2 w_{xx} = w_t$ and has the boundary conditions $w(0,t) = w(L,t) = 0$. The initial temperature distribution is found as follows $w(x,0) = u(x,0) - v(x) = 25 - 3x$. We must compute the sine series for $25 - 3x$.

$$c_n = \frac{2}{20}\int_0^{20} (25 - 3x)\sin\frac{n\pi x}{20}dx = \frac{10}{n\pi}(5 + 7(-1)^n).$$

The computation is similar to Problem 10 from 10.5. The general solution for $w(x,t)$ is the type studied in 10.5 therefore

$$w(x,t) = \sum_{n=1}^{\infty} \frac{10}{n\pi}(5 + 7(-1)^n)\ e^{-.86n^2\pi^2 t/20^2} \sin\frac{n\pi x}{20}.$$

and

$$u(x,t) = 3x + \sum_{n=1}^{\infty} \frac{10}{n\pi}(5 + 7(-1)^n)\ e^{-.86n^2\pi^2 t/20^2} \sin\frac{n\pi x}{20}.$$

d. *Determine how much time must elapse before the temperature at $x = 5cm$ comes (and remains) within 1 % of its steady-state value.*

We want to find the time t such that $|u(5,t) - v(5)| = .01v(5)$, or equivalently $|w(5,t)| = .15$.

$$w(5,t) = \sum_{n=1}^{\infty} \frac{10}{n\pi}(5 + 7(-1)^n)\ e^{-.86n^2\pi^2 t/20^2} \sin\frac{n\pi 5}{20} \cong \frac{-20}{\pi}\ e^{-.86\pi^2 t/20^2}\frac{\sqrt{2}}{2}$$

Taking absolute value, setting this to .15, and solving, we obtain $\ln\frac{.15\pi}{10\sqrt{2}} = -.86\pi^2 t/20^2$ implies $t \cong 160$. This approximation is reasonable because in this time range the exponential term is about e^{-3} for $n = 1$ and e^{-12} for $n = 2$.

10a. *Let the ends of a copper rod 100cm long be maintained at $0°C$. Suppose that the center of the bar is heated to $100°C$ by an external heat source and that this situation is maintained until a steady state results. Find this steady-state temperature distribution.*

Because the center of the bar is kept at 100°C, external heat is being applied and the differential equation is does not hold here. Instead the problem reduces to two separate boundary-value problems on the rod from 0 to 50 cm and on the rod from 50 to 100cm. For the rod from 0 to 50 cm, $T_1 = 0$, $T_2 = 100$, and $L = 50$ therefore

$$v(x) = (T_2 - T_1)\frac{x}{L} + T_1 = 2x, \quad 0 \le x \le 50.$$

For the rod from 50 to 100 cm, $T_1 = 100$, $T_2 = 0$, and $L = 50$ therefore

$$v(x) = (T_2 - T_1)\frac{x - 50}{L} + T_1 = 200 - 2x, \quad 50 \le x \le 100.$$

The $x - 50$ appears because we want x to start at 50 rather than 0 for this part of the rod.