Section 10.5

In Problems 1 and 3, determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

1. \( xu_{xx} + u_t = 0 \)
   
   This is separable. We try \( u(x, t) = X(x)T(t) \) and get \( u_{xx} = X''(x)T(t) \) and \( u_t = X(x)T'(t) \). Substituting into the equation we get \( xX''T = XT' \), so \( xX'' = -\lambda X \) and \( T' = -\lambda T \). From these we get the ordinary differential equations:
   
   \[
   xX'' + \lambda X = 0 \\
   T' + \lambda T = 0
   \]

3. \( u_{xx} + u_{xt} + u_t = 0 \)
   
   This is separable. We try \( u(x, t) = X(x)T(t) \) and substitute into the equation we get \( X''T + X'T + XT' = 0 \), which implies \( xX'' = -\lambda X \) and \( T' = -\lambda T \). From these we get the ordinary differential equations:
   
   \[
   X'' + \lambda X + \lambda X = 0 \\
   T' - \lambda T = 0
   \]

7. Find the solution of the heat conduction problem

   \[
   100u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0; \\
   u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0; \\
   u(x, 0) = \sin 2\pi x - \sin 5\pi x, \quad 0 \leq x \leq 1;
   \]

   The general solution to the heat equation with boundary conditions of this form is

   \[
   u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin \frac{n\pi x}{L}.
   \]

   where the \( c_n \) are determined uniquely by

   \[
   u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}
   \]

   In this case \( L = 1 \) and \( \alpha^2 = 100 \). \( u(x, 0) \) is already in the form of a sine series, where \( c_2 = 1 \), \( c_5 = -1 \), and the other \( c_n \) are 0. Therefore

   \[
   u(x, t) = c_2 e^{-2^2\pi^2 100 t/1} \sin \frac{2\pi x}{1} + c_5 e^{-5^2\pi^2 100 t/1} \sin \frac{5\pi x}{1} = e^{-400\pi^2 t} \sin(2\pi x) - e^{-2500\pi^2 t} \sin(5\pi x).
   \]

   Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all \( t > 0 \). In problems 9 and 10 find an expression for the temperature \( u(x, t) \) if the initial temperature distribution in the rod is the given function. Suppose that \( \alpha^2 = 1 \).
9. \(u(x, 0) = 50, \; 0 < x < 40\).

The sine series for the function \(u(x, 0)\) is computed as follows:

\[
 c_n = \frac{2}{L} \int_0^L u(x, 0) \sin \frac{n\pi x}{L} \, dx = \frac{100}{n\pi} \left( \cos \frac{n\pi}{L} \right)_{0}^{L} = \frac{100}{n\pi} (\cos(n\pi) - 1) = \begin{cases} \frac{200}{n\pi} & \text{n odd} \\ 0 & \text{n even} \end{cases}
\]

Plugging this into the general solution with \(L = 40\) yields

\[
u(x, t) = \sum_{n \text{ odd}} \frac{200}{n\pi} e^{-n^2\pi^2t/40^2} \sin \frac{n\pi x}{40}.
\]

10. \(u(x, 0) = \begin{cases} x & 0 \leq x \leq 20, \\ 40 - x & 20 \leq x \leq 40 \end{cases} \)

The sine series for the function \(u(x, 0)\) is computed as follows:

\[
 c_n = \frac{2}{L} \int_0^{L/2} x \sin \frac{n\pi x}{L} \, dx + \frac{2}{L} \int_{L/2}^{L} (L - x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \left( \frac{-Lx}{n\pi} \cos \frac{n\pi x}{L} \right)_{0}^{L/2} + \frac{2}{L} \left( \frac{L(x - L)}{n\pi} \cos \frac{n\pi x}{L} \right)_{0}^{L/2} = \frac{2}{L} \left( \frac{L^2}{n^2\pi^2} \sin \frac{n\pi L/2}{L} \right) + \frac{2}{L} \left( - \frac{L^2}{n^2\pi^2} \sin \frac{n\pi L/2}{L} \right).
\]

For the last equality we used \(\cos \frac{n\pi L/2}{L} = 0\) and \(\sin \frac{n\pi L/2}{L} = 0\). Continuing the simplification, we have

\[
c_n = \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2}
\]

Plugging this into the general solution with \(L = 40\) yields

\[
u(x, t) = \sum_{n = 1}^{\infty} \frac{160}{n^2\pi^2} \sin \frac{n\pi}{2} e^{-n^2\pi^2t/40^2} \sin \frac{n\pi x}{40}.
\]

14. For the rod in Problem 9:

d. How long does it take for the entire rod to cool off to a temperature of no more than 1°C

Because of the symmetry of the initial temperature distribution and the boundary conditions, the warmest point is always at \(x = 20\). We want to find the time \(t\) such that \(u(20, t) = 1\)°C.

\[
u(20, t) = \sum_{n \text{ odd}} \frac{200}{n\pi} e^{-n^2\pi^2t/40^2} \sin \frac{n\pi 20}{40} \approx \frac{200}{\pi} e^{-\pi^2t/40^2}
\]

Setting this to 1 and solving, we obtain \(\ln \frac{\pi}{200} = -\pi^2t/40^2\) implies \(t \approx 673\). This is in seconds because \(\alpha^2\) has units \(cm^2/sec\) and \(L\) has units \(cm\). This approximation is reasonable because in this time range the exponential term is about \(e^{-4}\) for \(n = 1\) and \(e^{-36}\) for \(n = 3\).

15. For the rod in Problem 10:
d. How long does it take for the entire rod to cool off to a temperature of no more than \(1^\circ C\)

Because of the symmetry of the initial temperature distribution and the boundary conditions, the warmest point is always at \(x = 20\). We want to find the time \(t\) such that \(u(20, t) = 1^\circ C\).

\[
u(20, t) = \sum_{n=1}^{\infty} \frac{160}{n^2 \pi^2} \sin \frac{n\pi}{2} e^{-n^2 \pi^2 t/40^2} \sin \frac{n\pi 20}{40} \approx \frac{160}{\pi^2} \left( \sin \frac{\pi}{2} \right) e^{-\pi^2 t/40^2} \sin \frac{\pi 20}{40} = \frac{160}{\pi^2} e^{-\pi^2 t/40^2}
\]

Setting this to 1 and solving, we obtain \(\ln \frac{\pi^2}{160} = -\pi^2 t/40^2\) implies \(t \approx 452\). This approximation is reasonable because in this time range the exponential term is about \(e^{-3}\) for \(n = 1\) and \(e^{-12}\) for \(n = 2\).

19a. A silver rod 20 cm long is heated to a uniform temperature of 100\(^\circ\)C. At \(t = 0\) the ends of the bar are kept at 0\(^\circ\)C. Find an expression for the temperature at any point at any time \(t > 0\). Find how long will it take for the center to cool to 5\(^\circ\)C.

This is the same as Problem 9 except with different constants \(L, \alpha\), and initial temperature. For silver, \(\alpha^2 = 1.71\).

\[
u(x, t) = \sum_{n \text{ odd}} \frac{400}{n\pi} e^{-n^2 1.71 \pi^2 t/20^2} \sin \frac{n\pi x}{20}.
\]

\[
u(10, t) = \sum_{n \text{ odd}} \frac{400}{n\pi} e^{-n^2 1.71 \pi^2 t/20^2} \sin \frac{n\pi 10}{20} \approx \frac{400}{\pi} e^{-1.71 \pi^2 t/20^2}
\]

Setting this to 5 and solving, we obtain \(\ln \frac{\pi}{80} = -1.71 \pi^2 t/20^2\) implies \(t \approx 77\). This approximation is reasonable because in this time range the exponential term is about \(e^{-3}\) for \(n = 1\) and \(e^{-12}\) for \(n = 2\).

Section 10.6

In Problems 1 and 4 find the steady-state solution of the heat equation \(\alpha^2 u_{xx} = u_t\) that satisfies the given set of boundary conditions.

1. \(u(0, t) = 10, \quad u(50, t) = 40\)

The steady-state solution \(v(x)\) must satisfy the differential equation with \(v_t = 0\), which implies \(v_{xx} = 0\). Therefore \(v(x) = c_1 + c_2 x\), for some constants \(c_1\) and \(c_2\). \(v(x)\) must also satisfy the boundary conditions \(v(0) = 10\) and \(v(50) = 40\) so \(10 = v(0) = c_1\) and \(40 = v(50) = c_1 + 50c_2\). Thus \(c_2 = \frac{3}{5}\) and \(v(x) = 10 + \frac{3}{5}x\).

4. \(u_x(0, t) = 0, \quad u(L, t) = T\)

The steady-state solution \(v(x)\) must satisfy the differential equation with \(v_t = 0\), which implies \(v_{xx} = 0\). Therefore \(v(x) = c_1 + c_2 x\), for some constants \(c_1\) and \(c_2\). \(v(x)\) must also satisfy the boundary conditions \(v_x(0) = 0\) and \(v(L) = T\) so \(0 = v_x(0) = c_2\) and \(T = v(L) = c_1 + Lc_2 = c_1\). Thus \(v(x) = T\).

9. Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25\(^\circ\)C. Suppose that at time \(t = 0\) the end \(x = 0\) is cooled to 0\(^\circ\)C while the end \(x = 20\) is heated to 60\(^\circ\)C, and both are thereafter maintained at those temperatures.

a. Find the temperature distribution in the rod at any time.

In this problem \(L = 20, \quad T_1 = 0, \quad T_2 = 60\), and \(\alpha^2 = .86\). The steady-state solution is

\[
v(x) = (T_2 - T_1) \frac{x}{L} + T_1 = 3x.
\]
We will express \( u(x, t) \) as the sum of the steady-state solution and another solution \( w(x, t) \) that depends on time: \( u(x, t) = v(x) + w(x, t) \). As shown in the discussion in 10.6 (p. 614), \( w(x, t) \) also satisfies \( \alpha^2 w_{xx} = w_t \) and has the boundary conditions \( w(0, t) = w(L, t) = 0 \). The initial temperature distribution is found as follows \( w(x, 0) = u(x, 0) - v(x) = 25 - 3x \).

We must compute the sine series for \( 25 - 3x \).

\[
\begin{align*}
c_n &= \frac{2}{20} \int_0^{20} (25 - 3x) \sin \frac{n\pi x}{20} dx = \frac{10}{n\pi} (5 + 7(-1)^n).
\end{align*}
\]

The computation is similar to Problem 10 from 10.5. The general solution for \( w(x, t) \) is the type studied in 10.5 therefore

\[
\begin{align*}
w(x, t) &= \sum_{n=1}^{\infty} \frac{10}{n\pi} (5 + 7(-1)^n) e^{-\frac{86n^2\pi^2t}{20^2}} \sin \frac{n\pi x}{20},
\end{align*}
\]

and

\[
\begin{align*}
u(x, t) &= 3x + \sum_{n=1}^{\infty} \frac{10}{n\pi} (5 + 7(-1)^n) e^{-\frac{86n^2\pi^2t}{20^2}} \sin \frac{n\pi x}{20}.
\end{align*}
\]

d. **Determine how much time must elapse before the temperature at \( x = 5 \) cm comes (and remains) within 1\% of its steady-state value.**

We want to find the time \( t \) such that \(|u(5, t) - v(5)| = 0.01v(5)\), or equivalently \(|w(5, t)| = 0.15\).

\[
\begin{align*}
w(5, t) &= \sum_{n=1}^{\infty} \frac{10}{n\pi} (5 + 7(-1)^n) e^{-\frac{86n^2\pi^2t}{20^2}} \sin \frac{n\pi 5}{20} = -\frac{20}{\pi} e^{-\frac{86\pi^2t}{20^2}} \sqrt{2}
\end{align*}
\]

Taking absolute value, setting this to 0.15, and solving, we obtain \( \ln \frac{15\pi}{10\sqrt{2}} = -0.86\pi^2t/20^2 \) implies \( t \approx 160 \). This approximation is reasonable because in this time range the exponential term is about \( e^{-3} \) for \( n = 1 \) and \( e^{-12} \) for \( n = 2 \).

10a. **Let the ends of a copper rod 100 cm long be maintained at 0°C. Suppose that the center of the bar is heated to 100°C by an external heat source and that this situation is maintained until a steady state results. Find this steady-state temperature distribution.**

Because the center of the bar is kept at 100°C, external heat is being applied and the differential equation is not held here. Instead the problem reduces to two separate boundary-value problems on the rod from 0 to 50 cm and on the rod from 50 to 100 cm. For the rod from 0 to 50 cm, \( T_1 = 0 \), \( T_2 = 100 \), and \( L = 50 \) therefore

\[
\begin{align*}
v(x) &= (T_2 - T_1) \frac{x}{L} + T_1 = 2x, \quad 0 \leq x \leq 50.
\end{align*}
\]

For the rod from 50 to 100 cm, \( T_1 = 100 \), \( T_2 = 0 \), and \( L = 50 \) therefore

\[
\begin{align*}
v(x) &= (T_2 - T_1) \frac{x - 50}{L} + T_1 = 200 - 2x, \quad 50 \leq x \leq 100.
\end{align*}
\]

The \( x - 50 \) appears because we want \( x \) to start at 50 rather than 0 for this part of the rod.