Math 54 Spring 2005

Solutions to Homework Section 3.2

April 8th, 2005

1. *Find the Wronskian of the functions* e^{2t} *and* $e^{-3t/2}$.

   Det $\begin{pmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{pmatrix} = e^{2t}(-\frac{3}{2}e^{-3t/2}) - e^{-3t/2}(2e^{2t}) = -\frac{7}{2}e^{t/2}$.

2. *Find* $W(\cos t, \sin t)(t)$.

   $\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$.

3. *Find* $W(e^{-2t}, te^{-2t})(t)$.

   $\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(-2t+1) \end{vmatrix} = e^{-4t}(-2t+1) - -2te^{-4t} = e^{-4t}$.

4. *Find* $W(x, xe^x)(x)$.

   $\begin{vmatrix} x & xe^x \\ 1 & e^x(x+1) \end{vmatrix} = xe^x(x+1) - xe^x = x^2e^x$.

5. *Find* $W(e^t \sin t, e^t \cos t)(t)$.

   $\begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = e^{2t}(\sin t \cos t - \sin^2 t) - e^{2t}(\sin t \cos t + \cos^2 t) = -e^{2t}$.

6. *Find* $W(\cos^2\theta, 1 + \cos 2\theta)(x)$.

   $\begin{vmatrix} \cos^2\theta & 1+\cos 2\theta \\ -2\cos\theta\sin\theta & -2\sin 2\theta \end{vmatrix} = -2\cos^2\theta\sin 2\theta + 2(1+\cos 2\theta)\cos\theta\sin\theta = -2\cos^2\theta\sin 2\theta + 2(2\cos^2\theta)\cos\theta\sin\theta = 2\cos^2\theta(-\sin 2\theta + 2\cos\theta\sin\theta) = 0$.

   In problems 7 through 12 determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7. $ty'' + 3y = t$, $y(1) = 1, y'(1) = 2$.

   To apply theorem 3.2.1 we have to divide the ODE by t, obtaining $y'' + \frac{3}{t}y = 1$. Using the notation of theorem 3.2.1 we have $p(t) = 0, q(t) = \frac{3}{t}, g(t) = 1$. $q(t)$ is not defined at zero. So the largest interval containing 1 on which p, q and g are defined and continuous is $(0, \infty)$, and theorem 3.2.1 ensures the existence and uniqueness on this interval of a solution satisfying the initial conditions.

8. $(t-1)y'' - 3ty' + 4y = \sin t$, $y(-2) = 2, y'(-2) = 1$.

   To apply theorem 3.2.1 we have to divide the ODE by $t-1$, obtaining $y'' - \frac{3t}{t-1}y' + \frac{4}{t-1}y = \frac{\sin t}{t-1}$. Using the notation of theorem 3.2.1 we have $p(t) = -\frac{3t}{t-1}, q(t) = \frac{4}{t-1}, g(t) = \frac{\sin t}{t-1}$. $p(t)$, $q(t)$, and $g(t)$ are not defined at 1. So the largest interval containing -2 on which p, q and g are defined and continuous is $(-\infty, 1)$, and theorem 3.2.1 ensures the existence and uniqueness on this interval of a solution satisfying the initial conditions.

18. *If the Wronskian W of f and g is t^2e^t and if $f(t) = t$, find $g(t)$.*

    We have W=Det $\begin{pmatrix} t & g(t) \\ 1 & g'(t) \end{pmatrix}$=$tg' - g$. On the other side we know that $W = t^2e^t$, so g must satisfy $tg' - g = t^2e^t$. This implies $g' - g/t = te^t$ for $t \neq 0$ and $g(0) = 0$ (this is seen by evaluating the equation at $t = 0$).
    Now we have to solve this ODE, for example using the integrating factor $e^{\int_0^t -1/x dx} = e^{-\ln t} = 1/t$. We multiply by the integrating factor to obtain $g'/t - g/t^2 = e^t$, which yields $(g/t)' = e^t$. Integrating both sides with respect to t we obtain $g/t = e^t + c$ for any constant c. We obtain $g = t(e^t + c)$ and note that it satisfies $g(0) = 0$, so it's a solution for all t.

22. *Find the fundamental set of solutions specified by Theorem 3.2.5 for $y'' + 4y' + 3y = 0$ with initial point $t_0 = 1$.*

    The associated characteristic equation to this differential equation is $r^2 + 4r + 3 = 0$, which has solutions $r = -1, -3$. Therefore, the general solution to the differential equation is $y = c_1e^{-t} + c_2e^{-3t}$. Since we'll soon need it, let's note that its derivative is $y' = -c_1e^{-t} - 3c_2e^{-3t}$.

    For the set of solutions y_1 and y_2 specified by Theorem 3.2.5 (with $t_0 = 1$), we first obtain y_1 by letting $y(1) = 1$ and $y'(1) = 0$ in our general solution. This gives us the system of equations

    $$\begin{aligned} c_1e^{-1} + c_2e^{-3} &= 1 \\ -c_1e^{-1} - 3c_2e^{-3} &= 0 \end{aligned}$$

    It follows that $c_1 = 3/2e$ and $c_2 = -e^3/2$, giving us the solution $y_1 = \frac{3e}{2}e^{-t} - \frac{e^3}{2}e^{-3t}$. To obtain y_2, we let $y(1) = 0$ and $y'(1) = 1$ in our general solution. This gives us the system of equations

    $$\begin{aligned} c_1e^{-1} + c_2e^{-3} &= 0 \\ -c_1e^{-1} - 3c_2e^{-3} &= 1 \end{aligned}$$

    It follows that $c_1 = e/2$ and $c_2 = -e^3/2$, giving us $y_2 = \frac{e}{2}e^{-t} - \frac{e^3}{2}e^{-3t}$. The y_1 and y_2 we have just found make up the fundamental set of solutions specified by Theorem 3.2.5.

24. *Verify that $y_1(t) = e^t$ and $y_2(t) = te^t$ are solutions of $y'' - 2y' + y = 0$ for $x \in \mathbb{R}$. Do they constitute a fundamental set of solutions?*

    For y_1 we have $y_1{}' = y_1{}'' = e^t$. Substituting this into the given differential equation gives us

    $$e^t - 2e^t + e^t = 0.$$

    For y_2 we have $y_2{}' = e^t(t + 1)$ and $y_2{}'' = e^t(t + 2)$. Substituting this into the given differential equation gives us

    $$e^t(t + 2) - 2e^t(t + 1) + te^t = 0.$$

    So y_1 and y_2 are both solutions to the differential equation. To see that they constitute a fundamental set of solutions, we examine their Wronskian:

    $$W(e^t, te^t) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t(t+1) \end{vmatrix} = e^{2t}(t + 1) - te^{2t} = e^{2t}$$

Since e^{2t} is non-zero for $x \in \mathbb{R}$ (the given interval), it follows by Theorem 3.2.4 that y_1 and y_2 form a fundamental set of solutions. Thus the general solution to the given differential equation is $y = c_1 e^t + c_2 t e^t$.

26. *Verify that* $y_1(t) = x$ *and* $y_2(t) = \sin x$ *are solutions of* $(1 - x\cot x)y'' - xy' + y = 0$ *for* $x \in (0, \pi)$. *Do they constitute a fundamental set of solutions?*

For y_1 we have ${y_1}' = 1$ and ${y_1}'' = 0$. Substituting this into the given differential equation gives us

$$0 - x + x = 0.$$

For y_2 we have ${y_2}' = \cos x$ and ${y_2}'' = -\sin x$. Substituting this into the given differential equation gives us

$$(1 - x\cot x)(-\sin x) - x\cos x + \sin x = 0.$$

So y_1 and y_2 are both solutions to the differential equation. To see that they constitute a fundamental set of solutions, we examine their Wronskian:

$$W(x, \sin x) = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix} = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x.$$

Since this is non-zero at $\pi/2$, which is in the interval $(0, \pi)$, it follows by Theorem 3.2.4 that y_1 and y_2 form a fundamental set of solutions (we need to divide by $(1 - x\cot x)$ to apply Theorem 3.2.4). Thus the general solution to the given differential equation is $y = c_1 x + c_2 \sin x$.