Math 54 Spring 2005

Solutions to Homework Section 5.4

April 4th, 2005

In Exercises 1-14, for the given $n \times n$ symmetric matrix A:
(a) Find any n linearly independent eigenvectors and verify that those associated with distinct eigenvalues are orthogonal, and
(b) Find an orthogonal matrix Q and a diagonal matrix Λ such that $Q^{-1}AQ = \Lambda$.

3. $A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$.

   [a.] The characteristic polynomial is $\det(\lambda I - A) = (\lambda - 6)(\lambda - 1)$ and so the eigenvalues are 6 and 1. Looking at $6I - A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ which is row equivalent to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, we see that a basis for $NS(6I - A)$ is $\{(2, -1)\}$, and so $(2, -1)$ is an eigenvector corresponding to 6. Looking at $I - A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$, we see similarly that $(1, 2)$ is an eigenvector with eigenvalue 1. $(2, -1)$ and $(1, 2)$ are clearly linearly independent. Also, $(2, -1) \cdot (1, 2) = 2 - 2 = 0$, and so $(2, -1)$ and $(1, 2)$ are orthogonal.

   [b.] After normalizing the vectors in (a), let $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$. Then Q is an orthogonal matrix, Λ is a diagonal matrix, and $Q^{-1}AQ = \Lambda$. This follows from various results in previous sections.

6. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   [a.] The characteristic polynomial is $\det(\lambda I - A) = (\lambda - 1)(\lambda + 1)$ and so the eigenvalues are 1 and -1. Looking at $I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ we see that a basis for $NS(I - A)$ is $\{(1, 1)\}$, and so $(1, 1)$ is an eigenvector corresponding to 1. Looking at $-I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, we see that $(1, -1)$ is an eigenvector with eigenvalue -1. $(1, 1)$ and $(1, -1)$ are clearly linearly independent. Also, $(1, 1) \cdot (1, -1) = 0$, and so $(1, 1)$ and $(1, -1)$ are orthogonal.

   [b.] After normalizing the vectors in (a), let $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then Q is an orthogonal matrix, Λ is a diagonal matrix, and $Q^{-1}AQ = \Lambda$.

8. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   [a.] The characteristic polynomial is $\lambda(\lambda^2 - 2)$ so the eigenvalues are 0, $\sqrt{2}$, and $-\sqrt{2}$. A basis for $NS(-A)$ is $\{(1, 0, -1)\}$, so this is an eigenvector corresponding to 0. A basis for $NS(\sqrt{2}I - A)$ is $\{(1, \sqrt{2}, 1)\}$, so this is an eigenvector corresponding to $\sqrt{2}$. A basis for $NS(-\sqrt{2}I - A)$ is $\{(1, -\sqrt{2}, 1)\}$, so this is an eigenvector corresponding to $-\sqrt{2}$.

We compute $(1, 0, -1) \cdot (1, \sqrt{2}, 1) = 0$, $(1, 0, -1) \cdot (1, -\sqrt{2}, 1) = 0$, and $(1, \sqrt{2}, 1) \cdot (1, -\sqrt{2}, 1) = 1 - 2 + 1 = 0$. All three vectors are orthogonal to each other. Therefore they are automatically linearly independent and form an orthogonal basis.

[b.] Normalizing the bases, let $Q = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix}$.

Then Q is an orthogonal matrix, Λ is a diagonal matrix, and $Q^{-1}AQ = \Lambda$.

13. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

[a.] By inspection, we see that $(1, -1, 0, 0)$, $(1, 0, -1, 0)$, and $(1, 0, 0, -1)$ are eigenvectors with eigenvalue 0, and $(1, 1, 1, 1)$ is an eigenvector with eigenvalue 4. Applying the Gram-Schmidt process (without normalizing) to these three eigenvectors with eigenvalue 0 yields the orthogonal vectors $(1, -1, 0, 0)$, $(1/2, 1/2, -1, 0)$, and $(1/3, 1/3, 1/3, -1)$. We will use these vectors as a basis for the eigenspace corresponding to eigenvalue 0. An easy calculation shows that these three vectors are orthogonal to $(1, 1, 1, 1)$, therefore all four vectors are linearly independent. In particular, the eigenvectors associated with distinct eigenvalues are orthogonal.

[b.] Normalizing this orthogonal basis, we let $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/2 \\ 0 & 0 & -3/\sqrt{12} & 1/2 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Then Q is an orthogonal matrix, Λ is a diagonal matrix, and $Q^{-1}AQ = \Lambda$.

In Exercises 15-20, eigenvectors and corresponding eigenvalues of a symmetric matrix A are given. Find a diagonal matrix Λ and an orthogonal matrix Q such that $A = Q\Lambda Q^{-1}$. You do not need to find A.

18. 5, $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$; -1, $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$; 5, $\begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$.

Calculating, $(3, 4, 5) \cdot (4, -3, 0) = 12 - 12 = 0$, $(3, 4, -5) \cdot (4, -3, 0) = 12 - 12 = 0$, and $(3, 4, 5) \cdot (3, 4, -5) = 9 + 16 - 25 = 0$, we see that all three eigenvectors given are orthogonal to each other. So after normalizing them, we take $Q = \frac{1}{5}\begin{bmatrix} 3/\sqrt{2} & 4 & 3/\sqrt{2} \\ 4/\sqrt{2} & -3 & 4/\sqrt{2} \\ 5/\sqrt{2} & 0 & -5/\sqrt{2} \end{bmatrix}$, and $\Lambda = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then Q is an orthogonal matrix, Λ is a diagonal matrix, and $A = Q\Lambda Q^{-1}$.

21. Show that if there is an orthogonal matrix Q that diagonalizes A, then A is symmetric.

Suppose that there is an orthogonal matrix Q that diagonalizes A. Then by definition, $Q^{-1}AQ =$

Λ for some diagonal matrix Λ. Multiplying by Q on the left, and by Q^{-1} on the right, we have $A = Q\Lambda Q^{-1}$. Then $A^T = (Q\Lambda Q^{-1})^T = (Q^{-1})^T \Lambda^T Q^T$. Since Λ is diagonal, $\Lambda^T = \Lambda$. Since Q is orthogonal, $Q^T = Q^{-1}$ and $(Q^{-1}))^T = (Q^T)^T = Q$. So $A^T = (Q^{-1})^T \Lambda^T Q^T = Q\Lambda Q^{-1} = A$. So $A = A^T$. Therefore A is symmetric.

22. A complex number $z = a + bi$ is real if and only if $z = \overline{z}$.

$a + bi = z = \overline{z} = a - bi \Leftrightarrow bi + bi = a - a \Leftrightarrow bi = 0 \Leftrightarrow b = 0 \Leftrightarrow z$ is real.

Jordan Normal Form

1. $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$. The characteristic polynomial is $(\lambda - 3)^2$. Because $A \neq 3I$, the only remaining possibility for the Jordan normal form is $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. We write A as $D + N$ where D is the diagonal matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and N is the nilpotent matrix $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. As in the notes, let $S = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$, the matrix with its first column the second column of N and its second column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $N = S \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S^{-1}$. Because $ND = DN$, this implies $A = D + N = S(3I)S^{-1} + S \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S^{-1} = SBS^{-1}$.

   A direct calculation shows $B^5 = \begin{bmatrix} 3^5 & 5 \cdot 3^4 \\ 0 & 3^5 \end{bmatrix}$. $A^5 = SB^5S^{-1} = \begin{bmatrix} 8 \cdot 3^4 & -5 \cdot 3^4 \\ 5 \cdot 3^4 & -2 \cdot 3^4 \end{bmatrix}$.

2. Because $BN = NB$, the binomial formula applies. For any nonnegative integer n,

$$(B + N)^n = \sum_{k=0}^{n} \binom{n}{k} N^k B^{n-k} .$$

   $N^2 = 0$ so all the terms for $k > 1$ are 0. We obtain

$$(B + N)^n = \binom{n}{0} B^n + \binom{n}{1} NB^{n-1} = B^n + nNB^{n-1}.$$

   Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.

$$p(B + N) = a_n(B + N)^n + a_{n-1}(B + N)^{n-1} + \ldots + a_1(B + N) + a_0 I$$

$$= a_n(B^n + nNB^{n-1}) + a_{n-1}(B^{n-1} + (n - 1)NB^{n-2}) + \ldots + a_1(B^1 + NB^0) + a_0 I$$

   by the result just proved. Now rearrange terms and factor out an N as follows:

$$= (a_n B^n + a_{n-1}B^{n-1} + \ldots + a_1 B + a_0 I) + (a_n nNB^{n-1} + a_{n-1}(n-1)NB^{n-2} + \ldots + a_1 N) = p(B) + Np'(B).$$

3. A generalization of the binomial formula is $(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$ for real n and $|x| < 1$ ($\binom{n}{k} = n(n-1)\ldots(n-k+1)/k!$). Motivated by this, and using the fact that $N^2 = 0$ as we did in problem 2, we guess that $(B + N)^n = B^n + nNB^{n-1}$ holds for rational numbers n. Let us check this guess in some special cases.

[a.] If $n=-1$,

$$(B+N)(B^{-1}-NB^{-2}) = BB^{-1}+NB^{-1}-BNB^{-2}-NNB^{-2} = I-N^2B^{-2} = I.$$

Therefore $(B^{-1}-NB^{-2}) = (B+N)^{-1}$. Here we have used the fact that $BN=NB$ implies $B^{-1}N = NB^{-1}$. Proof: $B^{-1}N = B^{-1}NBB^{-1} = B^{-1}BNB^{-1} = NB^{-1}$.

[b.] If $n=1/2$, we need to be more careful because it is not true in general that $B^{1/2}N = NB^{1/2}$. If this holds, we have

$$(B^{1/2}+\frac{1}{2}NB^{-1/2})^2 = (B^{1/2})^2 + 2\frac{1}{2}NB^{1/2}B^{-1/2} + \frac{1}{4}N^2(B^{-1/2})^2$$

$$= B+N+\frac{1}{4}N^2(B^{-1/2})^2 = B+N.$$

Therefore $B^{1/2}+\frac{1}{2}NB^{-1/2}$ is a square root of $B+N$ provided $B^{1/2}N = NB^{1/2}$.

[c.] Define e^A to be $\sum_{k=0}^{\infty} A^k/k!$. This sum converges for any matrix A, but we won't check that here.

$$e^{B+N} = \sum_{k=0}^{\infty}(B+N)^k/k! = \sum_{k=0}^{\infty}(B^k + kNB^{k-1})/k!$$

using our formula from 2. Now separate this into two sums:

$$= \sum_{k=0}^{\infty}(B^k)/k! + \sum_{k=0}^{\infty}kNB^{k-1}/k! = e^B + N\sum_{k=1}^{\infty}kB^{k-1}/k!.$$

In the last step we delete the $k=0$ term because it is 0. Now use the fact that $k/k! = 1/(k-1)!$ for $k\geq 1$ to obtain

$$= e^B + N\sum_{k=1}^{\infty}B^{k-1}/(k-1)! = e^B + Ne^B = e^B(1+N).$$

Thus $e^A = e^B(1+N)$.

4. $A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$. The characteristic polynomial of A is $(\lambda-1)^2$. Therefore we can write A as $D+N$ where D is the diagonal matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and N is the nilpotent matrix $\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$. The notes show that $N^2=0$ in this situation. D and N commute ($DN=ND$) because $D=I$. Also, D is invertible, so we can apply problem 3.

$$A^{-1} = D^{-1}-ND^{-2} = I-N = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$\sqrt{A}$ is not unique because $\sqrt{D}$ is not unique. There are actually infinitely many square roots of $\sqrt{D}$. The square roots of D are the matrices similar to one of the following:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $D=I$, we obtain one possibility for the square root of A.

$$A^{1/2} = D^{1/2}+\frac{1}{2}ND^{-1/2} = I+\frac{1}{2}N = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

Another possibility for $\sqrt{A}$ is $\begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ does not commute with N and does not give a square root of A.

We can use problem 2 to compute the 5th power of the matrix in problem 1:

$$\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}^5 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}^5 + 5 \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}^4 \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^5 & 0 \\ 0 & 3^5 \end{bmatrix} + \begin{bmatrix} 5 \cdot 3^4 & -5 \cdot 3^4 \\ 5 \cdot 3^4 & -5 \cdot 3^4 \end{bmatrix} = \begin{bmatrix} 8 \cdot 3^4 & -5 \cdot 3^4 \\ 5 \cdot 3^4 & -2 \cdot 3^4 \end{bmatrix}.$$

This agrees with our computation in problem 1.