Math 54 Spring 2005

Solutions to Homework Section 7.6-7.8, B+D

April 22, 2005

Section 7.6

1. Find the general solution to $\mathbf{X}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

   SOLUTION: Consider the eigenspace $\mathbf{W}_{1+2i} = \text{Span}\{(1, -1+i) = (1,-1)+i(0,1)\}$. From this we derive the general solution: $\mathbf{X} = \mathbf{c}_1[e^t(\cos(2t)(1,-1) - \sin(2t)(0,1))] + \mathbf{c}_2[e^t(\sin(2t)(1,-1) + \cos(2t)(0,1))]$. The general solution spirals outward counterclockwise.

2. Find the general solution. $\mathbf{X}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\mathbf{X}$.

   SOLUTION: Consider the eigenspace $\mathbf{W}_{-1+2i} = \text{Span}\{(2,-i) = (2,0) + i(0,-1)\}$ From this we derive the general solution: $\mathbf{X} = \mathbf{c}_1[e^{-t}(\cos(2t)(2,0) - \sin(2t)(0,-1))] + \mathbf{c}_2[e^{-t}(\sin(2t)(2,0) + \cos(2t)(0,-1))]$. The general solution spirals inward clockwise.

3. $\mathbf{X}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\mathbf{X}$.

   SOLUTION: Consider the eigenspace $\mathbf{W}_i = \text{Span}\{(5, 2-i) = (5,2) + i(0,-1)\}$. From this we derive the general solution: $\mathbf{X} = c_1(\cos(t)(5,2) - \sin(t)(0,-1)) + \mathbf{c}_2(\sin(t)(5,2) + \cos(t)(0,-1))$. The general solution is a counterclockwise orbit.

4. $\mathbf{X}' = \begin{bmatrix} 2 & -5/2 \\ 9/5 & -1 \end{bmatrix}$.

   SOLUTION: Consider the eigenspace $\mathbf{W}_{1/2(1+3i)} = \text{Span}\{(5, 3+3i) = (5,3) + i(0,3)\}$. From this we derive the general solution: $\mathbf{X} = c_1 e^{t/2}(\cos(3t/2)(5,3) - \sin(3t/2)(0,3)) + c_2(\sin(3t/2)(5,3) + \cos(3t/2)(0,3))$. The general solution spirals outward counterclockwise.

9. Find the solution to the following initial value problem. Describe the solution as $t \to \infty$.

   $$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\mathbf{x} \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   SOLUTION: The eigenvalues for this matrix are $-1-i$ and $-1+i$. Corresponding eigenvectors are

   $$\begin{pmatrix} 2-i \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

   The general solution is therefore

   $$\mathbf{x}(t) = a\begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{(-1-i)t} + b\begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t}$$

   Substituting $t = 0$ we see that $a = b = 1/2$. Using Euler's formula to expand out the complex exponential and simplifying we get the particular solution

   $$\mathbf{x}(t) = e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$$

Because the exponential is decreasing as $t \to \infty$ we see that the solution spirals into the origin. Looking at the form of the solution we also see that it spirals in counterclockwise.

Section 7.8

1. Find the general solution of the system and sketch a phase portrait.

   SOLUTION: We are asked to solve $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$.

$$det(A - rI) = \begin{vmatrix} 3 - r & -4 \\ 1 & -1 - r \end{vmatrix} = r^2 - 2r + 1 = (r - 1)^2$$

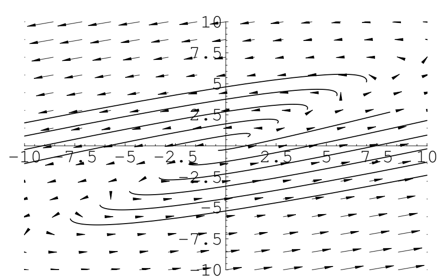
   So we have only one eigenvalue $r = 1$. Also we can check that the eigenspace is 1-dimensional and that $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector.

   So we must find $\mathbf{w}$ with $(A - I)\mathbf{w} = \mathbf{v}$. An easy computation shows that $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is such a vector.

   Now we merely write down the general solution:

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (t e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

   The phase portrait superimposed on the direction field is given below: trajectories go in the direction indicated by the direction field. The line through the eigenvector above is an asymptote.



2. Find the general solution of the system and sketch a phase portrait.

   SOLUTION: We are asked to solve $\mathbf{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \mathbf{x}$.

$$det(A - rI) = \begin{vmatrix} 4 - r & -2 \\ 8 & -4 - r \end{vmatrix} = r^2$$

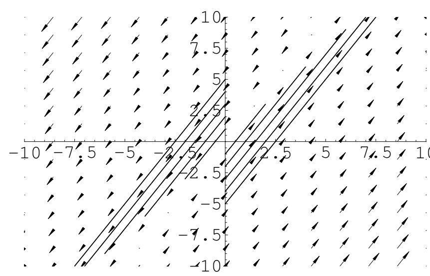
   So we have only one eigenvalue $r = 0$. Also we can check that the eigenspace is 1-dimensional and that $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector.

   So we must find $\mathbf{w}$ with $A\mathbf{w} = \mathbf{v}$. Another easy computation shows that $\mathbf{w} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$ is such a vector.

Now, noting that $e^{0t} = 1$ we again just write down the general solution:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 (t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix})$$

Again the phase portrait superimposed on the direction field is given below: all trajectories are straight lines which are parallel to the eigenvector; they swap direction at the line through the eigenvector.



3. $\mathbf{X}' = \begin{bmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{bmatrix} \mathbf{X}$.

   SOLUTION: We have $\mathbf{W}_{-1} = \text{Span}\{(2,1)\}$. One finds that $(0,2)$ is a generalized eigenvector. The general solution is thus $\mathbf{X}(t) = c_1(2,1)e^{-t} + c_2((2,1)te^{-t} + (0,2)e^{-t})$.

4. $\mathbf{X}' = \begin{bmatrix} -3 & 5/2 \\ -5/2 & 2 \end{bmatrix} \mathbf{X}$.

   SOLUTION: We have $\mathbf{W}_{-1/2} = \text{Span}\{(1,1)\}$. One finds that $(0,2/5)$ is a generalized eigenvector. The general solution is thus $\mathbf{X}(t) = c_1(1,1)e^{-t/2} + c_2((1,1)te^{-t/2} + (0,2/5)e^{-t/2})$.

7. $\mathbf{X}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{X}$, $\mathbf{X}(0) = (3,2)$.

   SOLUTION: The general solution is given by $\mathbf{X}(t) = c_1(4,4)e^{-3t} + c_2((4,4)te^{-3t} + (1,0)e^{-3t})$. The initial condition forces $c_1 = 1/2$ and $c_2 = 1$. Thus $X(t) = (3,2)e^{-3t} + (4,4)te^{-3t}$.