Math 54 Spring 2005

Solutions to Homework Section 7.5, B+D

April 18, 2005

1. $\mathbf{X}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{X}$.

   SOLUTION: The eigenspaces of the matrix are $\mathbf{W}_2 = \text{Span}\{\mathbf{v}_1 = (2,1)\}$ and $\mathbf{W}_{-1} = \text{Span}\{\mathbf{v}_2 = (1,2)\}$. The general solution is thus $\mathbf{X}(t) = c_1 e^{2t}\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2$. Any solution goes off to infinity asymptotically along the line defined by $\mathbf{v}_1$

2. $\mathbf{X}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \mathbf{X}$.

   SOLUTION: The eigenspaces of the matrix are $\mathbf{W}_{-1} = \text{Span}\{\mathbf{v}_1 = (1,1)\}$ and $\mathbf{W}_{-2} = \text{Span}\{\mathbf{v}_2 = (2,3)\}$. The general solution is thus $\mathbf{X}(t) = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-2t}\mathbf{v}_2$. Any solution approaches zero asymptotically along the line defined by $\mathbf{v}_1$.

5. $\mathbf{X}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{X}$.

   SOLUTION: The eigenspaces of the matrix are $\mathbf{W}_{-1} = \text{Span}\{\mathbf{v}_1 = (1,1)\}$ and $\mathbf{W}_{-3} = \text{Span}\{\mathbf{v}_2 = (1,-1)\}$. The general solution is thus $\mathbf{X}(t) = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-3t}\mathbf{v}_2$. Any solution approaches zero asymptotically along the line defined by $\mathbf{v}_1$.

14. $\mathbf{X}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{X}$.

    SOLUTION: The eigenspaces of the matrix are $\mathbf{W}_1 = \text{Span}\{\mathbf{v}_1 = (-1,4,1)\}$, $\mathbf{W}_3 = \text{Span}\{\mathbf{v}_2 = (1,2,1)\}$, and $\mathbf{W}_{-2} = \text{Span}\{\mathbf{v}_3 = (1,-1,-1)\}$. The general solution is thus $\mathbf{X}(t) = c_1 e^{t}\mathbf{v}_1 + c_2 e^{3t}\mathbf{v}_2 + c_3 e^{-2t}\mathbf{v}_3$.

16. $\mathbf{X}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \mathbf{X}$, $\mathbf{X}(0) = (1,3)$.

    SOLUTION: The eigenspaces of the matrix are $\mathbf{W}_3 = \text{Span}\{\mathbf{v}_1 = (1,5)\}$ and $\mathbf{W}_{-1} = \text{Span}\{(1,1)\}$. Thus, the general solution is $\mathbf{X}(t) = \mathbf{c}_1 e^{3t}\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2$. To find the unique solution satisfying the initial conditions, we solve: $(1,3) = \mathbf{X}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1(1,5) + c_2(1,1)$. A simple calculation shows $c_1 = c_2 = 1/2$.

18. $\mathbf{X}' = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathbf{X}$, $\mathbf{X}(0) = (7,5,5)$.

    SOLUTION: The eigenspaces of the matrix are $\mathbf{W}_4 = \text{Span}\{\mathbf{v}_1 = (2,1,-8)\}$, $\mathbf{W}_1 = \text{Span}\{\mathbf{v}_2 = (1,2,-1)\}$, and $\mathbf{W}_{-1} = \text{Span}\{\mathbf{v}_3 = (1,-2,1)\}$. The general solution is thus $\mathbf{X}(t) = c_1 e^{4t}\mathbf{v}_1 + c_2 e^{t}\mathbf{v}_2 + c_3 e^{-t}\mathbf{v}_3$. The initial condition $\mathbf{X}(0) = (7,5,5) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ forces $c_1 = -1$, $c_2 = 6$, and $c_3 = 3$.