Theorem Let A be a set and let $\tau_1, \tau_2, \ldots \tau_n$ be transpositions in S_A such that the product $\tau_1\tau_2\cdots\tau_n$ is the identity. Then n is even.

Proof: By induction on n. If n is zero, this is trivial For the induction step, we assume that the theorem is true for all $n' < n$ and that the above product is zero.

We shall repeatedly use the following trick:

Trick: If α and β are transpositions, then $\alpha\beta = \beta\alpha'$, where α' is another transposition.

Claim 1. If $\tau_1 \ldots \tau_n = e$, then there is a sequence of transpositions $\tau'_1, \ldots \tau'_n$ such that $\tau'_1 \cdots \tau'_n = e$ and such that for some $i < j$, $\tau'_i = \tau'_j$.

Claim 2. If $\tau_1, \ldots \tau_n = e$ and there exist $i < j$ such that $\tau_i = \tau_j$, then there is a sequence of transposition $\tau'_1, \ldots \tau'_{n-2}$ whose product is zero.

If we can prove both these claims, then the theorem follows, because the induction assumption applied to the new sequence in claim 2 tells us that $n-2$ is even, hence n is even.

Proof of claim 1. Choose an element a such that τ_1 moves a. Let S be the set of integers i such that τ_i does not move a and τ_{i+1} does move a. Suppose first that S is empty. Then we have

$$e = (ax_1)(ax_2)\cdots(ax_m)\tau_{m+1}\cdots\tau_n,$$

where τ_i does not move a if $i > m$. Since this product σ is the identity, $\sigma(a) = a$, and we see from the above formula that x_m must equal some x_i for $i < m$. This proves the claim in this case. If S is not empty, it has a smallest element, call it j. Then by the trick, we $\tau_j\tau_{j+1} = \tau_{j+1}\tau'$ for some transposition τ', so have $e = \tau_1, \cdots \tau_{j-1}\tau_{j+1}\tau'\tau_{j+2}\cdots$. Note that all the transpositions up to and including τ_{j+1}, which now in the jth place, move a. The minimum set of the set S for this list has to be at least $j+1$. Repeating this process, eventually S will become empty, and claim 1 will be proved.