1. Let $G$ be a cyclic group with only one generator. Then $G$ has at most two elements. To see this, note that if $g$ is a generator for $G$, then so is $g^{-1}$. If $G$ has only one generator, it must be the case that $g = g^{-1}$. But then $g^2 = e$. Since $g$ generates $G$, it follows that $G$ has at most two elements.

2. Let $H$ be a nonempty finite subset of a group $G$. Prove that if $H$ is closed, then it is a subgroup. To see this, suppose $h \in H$. Since $H$ is closed, $h^n \in H$ for every positive integer $n$, and since $H$ is finite, there exist $n, m$ with $n > m$ such that $h^n = h^m$. Then $h^{n-m} = e$, so $e \in H$. Furthermore, $h^{-1} = h^{n-m-1} \in H$. Thus $H$ is a subgroup.

3. Suppose $a$ and $b$ are elements of a group $G$ and that $ab$ has order $n$. Then $ba$ also has order $n$. To see this, note that $(ba)^n = a^{-1}(ab)^na$. Thus if $(ab)^n = e$, then $(ba)^n = e$. The converse holds by symmetry, and so $ab$ and $ba$ have the same order.

4. Suppose $G$ is a group which has only finitely many subgroups. We want to prove that $G$ is finite. First of all, recall that if $g \in G$, then $\langle g \rangle$ is a cyclic group containing $g$. If $\langle g \rangle$ is infinite, then $\langle g \rangle$ is isomorphic to $\mathbb{Z}$. But the group $\mathbb{Z}$ has infinitely many subgroups (one for each natural number), and then $G$ would also have infinitely many subgroups, a contradiction. Hence each $\langle g \rangle$ is finite. Since $G$ has only finitely many subgroups, and since each of these is finite, and since every element of $G$ is contained in a finite group, $G$ has only finitely many elements.

5. The group $V_4$ has the property that every proper subgroup is cyclic, but it itself is not cyclic.

6. Suppose that $G$ is a group and $a \in G$ is the unique element of order 2. Then $ax = xa$ for all $x \in G$. To see this, let $b := xax^{-1}$. Then $a = x^{-1}bx$. Since $a \neq e$, $b \neq a$, and an easy calculation shows that $b^2 = e$, so $\langle b \rangle$ has order 2. Since $a$ is unique, $a = b$, and this implies that $ax = xa$.

7. Let $G$ be a cyclic group of order $n$, written multiplicatively, and let $m$ be a divisor of $n$. Consider the set of all $x$ in $G$ such that $x^m = e$. 

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This is clearly a subgroup \( H \) of \( G \), since \( G \) is commutative. But every subgroup of \( G \) is cyclic, and if \( g \) generates \( G \), \( H = \langle g^d \rangle \) for some divisor \( d \) of \( n \). Since \( g^d \in H \), \( g^{dm} = e \), and hence \( n \) divides \( dm \). On the other hand, if \( d' := n/m \), \( g^{d'} \in H \), so \( d \) divides \( d' \). Write \( dm = an \) and \( d' = bd \). Then \( n = bmd = ban \), so \( b = a = 1 \). This implies that \( d = d' \), so \( H \) has \( m \) elements.