Mathematics 113 Professor A. Ogus
October 15, 2004

Algebra Midterm Exam

Note: There are five problems, on both sides of the page. Write your answers on the exam.

1. Definitions

   (a) If H is a subgroup of a permutation group S_A, what is the definition of an *orbit* of H on (or in) A?
   An orbit of H in A is a subset of A of the form $\{h(a) : h \in H\}$ for some element a of A.

   (b) What is the definition of a *cycle* in a permutation group?
   A cycle σ in a permutation group S_A is a permutation with the property that the cyclic subgroup generated by σ has just one nontrivial orbit in A.

2. Let A be a set and let H be a subgroup of the permutation group S_A. Prove that the set of orbits of H in A forms a partition of A.
   Let us define $R \subseteq A \times A$ to be the set of pairs (a, b) such that there exists an $h \in H$ with $b = h(a)$. We claim that R is an equivalence relation on A. Indeed, since H is a subgroup, $e \in H$, so $(a, a) \in R$ for any $a \in A$. Second, if $(a, b) \in R$ and $(b, c) \in R$, then there exist $h, h' \in H$ such that $b = h(a)$ and $c = h'(b)$, hence $c = (h'h)(a)$ and $(a, c) \in R$. Finally, if $(a, b) \in R$, then $b = h(a)$ for some h, hence $a = h^{-1}(a)$. Then from the definition of orbits, we see that the orbits are exactly the subsets corresponding to the partition defined by R.

3. Write the following permutation as a product of disjoint cycles, and compute its order and sign.

   $$(1\ 3\ 7\ 5)(2\ 4\ 3\ 5\ 6)(3\ 1\ 8\ 9\ 7)$$

   This is $(1\ 8\ 9\ 5\ 6\ 2\ 4\ 7)$, which has order 8 and is odd.

4. In the cyclic group $(\mathbf{Z}_{60}, +)$ of order 60 find:

   (a) The order and index of the subgroup generated by 4.
   Since $60 = 15 \cdot 4$, the element 4 has order 15, so this subgroup has order 15 and index 4.

   (b) The order and index of the subgroup generated by 18.
   Since $gcd(60, 18) = 6$, this is the same as the subgroup generated by 6, which has order 10 and index 6.

   (c) The order and index of the subgroup generated by 4 and 18. Is this group cyclic? If not, explain why not. If it is, find a generator.
   Yes, every subgroup of a cyclic subgroup is cyclic. Furthermore, this is the same as the subgroup generated by 6 and 4, which is the same as the subgroup generated by 2, so 2 is a generator. This group has order 30 and index 2.

5. Let G be a group of order 15. Suppose that G has only one subgroup of order 5 and only one subgroup of order 3.

   (a) How many elements of exact order 5 are there in G?
   The subgroup of order 5 must be cyclic, and contains all the elements of exact order 5, all of which are generators of this group. There are 4 such elements.

   (b) How many elements of exact order 3 are there in G?
   By the same reasoning, there are 2 such elements.

   (c) How many elements of exact order 2 are there in G?
   None, because 2 does not divide the order of G.

   (d) Bonus: Prove that G is cyclic.
   The only possibilities for orders of elements of G are 1, 3 and 5 and 15. The number of elements of orders 1, 3, and 5 are 1, 2, and 4, respectively. This gives a total of 7 elements, so there must be some elements of 15, so G is cyclic.