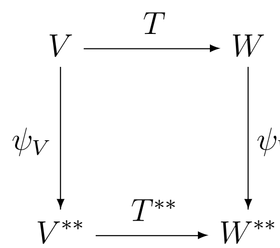
Solution to #11, section 2.6

This is mostly a matter of keeping track of the definitions. I think it is clearer to use a slightly different notation. Let V and W be vector spaces over F and let $T: V \to W$ be a linear transformation, and let $T^*: W^* \to V^*$ be its dual (transpose), defined by $T^*(g) := g \circ T$. We also have maps $\psi_V: V \to V^{**}$ and $\psi_W: W \to W^{**}$, where for example $\psi_V(v)(f) := f(v)$. The claim is that the following diagram commutes



We must prove that if v is any element of V, then $\psi_W(T(v)) = T^{**}(\psi_V(v))$. Both sides are functionals on W^*, and to prove that they are equal, it is enough to prove that they have the same value when evaluated at any element g of W^*. Let us start with the left side. At each stage, we just use the definition.

$$\psi_W(T(v))(g) \quad = \quad g(T(v))$$

Now the right side:

$$\begin{aligned} T^{**}(\psi_V(v))(g) &= (\psi_V(v) \circ T^*)(g) \\ &= \psi_V(v)(T^*(g)) \\ &= \psi_V(v)(g \circ T) \\ &= (g \circ T)(v) \\ &= g(T(v)) \end{aligned}$$

This is the same answer we got above, and concludes the proof.

Solution to number 19:

We do this in the finite dimensional case only. We know that there exists a basis $(v_1, \cdots v_n)$ for V and a $k \leq n$ such that $(v_1, \cdots, v_k)$ is a basis for W. Since W is a proper subspace of V, $k < n$. Let $(f_1, \cdots f_n)$ be the dual basis to $(v_1, \cdots v_n)$. Then $f_n(v_i) = 0$ for all $i \leq k$, and in particular for all the vectors in a basis for W. Hence $f_n(w) = 0$ for all $w \in W$. But $f_n(v_n) = 1$. Thus f_n does the job.

Solution to number 20:

Let $T\colon V \to W$ be a linear transformation.

(a). Suppose T is surjective. To prove that T^* is injective, it suffices to prove that its null space is zero. Suppose $T^*(g) = 0$. Then $g \circ T = 0$. This means that $g(T(v)) = 0$ for every $v \in V$. Since T is surjective, it follows that $g(w) = 0$ for every $w \in W$. This means that $g = 0$. thus, the null space of T^* is zero. Conversely, suppose that T is not surjective. Then its image (range) is a linear subspace W' of W. By the previous problem, there exists a $g \in W$ such that $g(w') = 0$ for all $w' \in W$, but $g \neq 0$. Then $g(T(v)) = 0$ for all $v \in V$, *i.e.*, $T^*(g) := g \circ T = 0$. But then T^* is not injective.

(b). Suppose T^* is surjective. For each nonzero $v \in V$, we prove that $T(v) \neq 0$. Since v is not zero, there exists an element $f \in V^*$ such that $f(v) \neq 0$. (This follows from the previous problem.) Since T^* is surjective, there exists some $g \in W^*$ such that $f = T^*(g) := g \circ T$. Since $f(v) = g(T(v))$ is not zero and g is linear, $T(v) \neq 0$. This shows that the null space of T is zero, so T is injective. Conversely, suppose that T is injective. Then it is an isomorphism onto its image, which is a linear subspace of W, and we may as well suppose that it is equal to this subspace. Then the problem amounts to showing that any linear functional on the subspace V of W extends to a linear functional on W. Let $(v_1, \ldots, v_n)$ be a basis for W such that $(v_1, \ldots, v_k)$ for a basis for V. Then any linear functional g on W is a linear combination of the restrictions of $(f_1, \ldots f_k)$ to W, and each of these extend, so so does g.