Mathematics 110 — Professor A. Ogus
November 6, 2003

Linear Algebra Midterm Exam

Write clearly, with complete sentences, explaining your work. You will be graded on clarity, style, and brevity. If you add false or irrelevant statements to a correct argument, you may lose points.

1. Answer the following questions using precise mathematical language.

   (a) If T is an operator on a vector space V, what is meant by a T-*invariant subspace* of V?
   A T-invariant subspace of V is a linear subspace W such that $T(w) \in W$ for every $w \in W$. (4 pts.)

   (b) What is the definition of the *rank* of a linear transformation?
   The rank of a linear transformation T is the dimension of its image. (4 pts.)

   (c) If T is a linear operator on a vector space V, what is meant by an *eigenvector* of T? By an *eigenvalue* of T?
   An eigenvector of T is a vector v such that $T(v)$ is a multiple of v. An eigenvalue of T is a scalar λ such that there exists a nonzero vector v such that $T(v) = \lambda v$. (6 pts.)

   (d) If V a finite dimensional vector space over F and T is an operator on V, what is meant by the *characteristic polynomial* of T? What is the key fact needed for this to make sense?
   The characteristic polynomial of T is $\det(T - \lambda I)$, regarded as a polynomial in the variable λ. To know this makes sense, we need to know that its value is independent of the basis used to compute the determinant. (6 pts.)

2. Let V be the space of infinitely differentiable functions $R \to R$. The map $D\colon f \mapsto f'$ defines a linear operator $V \to V$. For each natural number k, let v_k denote the function $x \mapsto e^{kx}$.

(a) Compute $D(v_k)$, and use your computation and a theorem we proved in class to show that the set of all v_k is linearly independent.
$D(v_k) = kv_k$. Thus, v_k is a nonzero eigenvector of D with eigenvalue k. Since the k's are distinct, a theorem proved in class implies that the set of all v_k is linearly independent. (10 pts.)

(b) Prove the theorem you used above in the special case at hand. That is, prove directly that the set of all v_k is linearly independent, using the method of proof the general theorem.
Suppose that S is a finite set of natural numbers and $a_k \in R$ for each $k \in S$ is a corresponding sequence of real numbers, such that $\sum_{k \in S} a_k v_k = 0$. We have to prove that necessarily each $a_k = 0$. Without loss of generality we may assume that $S = \{n : n \leq N\}$. The proof is by induction on N. If $N = 0$, the statement is clear, since $v_k \neq 0$. Assume the theorem is true for $N - 1$. Applying D to the equation $\sum_{k \leq N} a_k v_k = 0$ and using its linearity, we find that $\sum_{k \leq N} a_k k v_k = 0$. Multiplying the original equation by N, we find also that $\sum k \leq N a_k N v_k = 0$, and subtracting, we get $\sum_{k<N} a_k(N-k)v_k = 0$. Now the induction hypothesis implies that each $a_k(N-k) = 0$, and hence $a_k = 0$ for $k < N$. Then the original equation implies that $a_N v_N = 0$, hence $a_N = 0$. (20 pts.)

3. Let V be the vector space of functions from the reals to the reals. If $f \in V$, let $T(f)(x) := f''(x) + f(0)f(x)$. Let W be the T-cyclic subspace of V generated by x^2+1. Find a basis for W of the form $(v, Tv, \cdots T^{m-1}v)$. Using this basis, compute the characteristic polynomial of T_W.
Let $v = x^2 + 1$. We compuate that $T(v) = 2 + x^2 + 1 = 3 + x^2$ and $T^2(v) = 2+3(x^2+3) = 11+3x^2$. Then we notice that $T^2(v) = -v+4Tv$. So (v, Tv) is the desired basis. (10 pts.) In this basis, the matrix for T is $\begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}$, and so the characteristic polynomial of T is $\lambda^2 - 4\lambda + 1$. (10 pts.)

4. Say whether each of the following is true or false. In each case, explain your answer. In particular, if you claim that statement is true, give a

brief explanation, and if you claim it is false, give a counterexample. In each case, V is a finite dimensional vector space over a field.

(a) If T is an operator on V, every $v \in V$ generates a T-cyclic subspace of V.
This is true, just take the span of the set of all $T^k(v)$ with k a natural number. (6 pts).

(b) If T is an operator on V, V is T-cyclic.
This is false. For example, if $V = R^2$ and T is the zero transformation, V is not cyclic. (6 pts.)

(c) If T is an operator on V and if the characteristic polynomial of T splits, then T is diagonalizable.
This is false. For example, if V is R^2 and T is left multiplication by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, T is not diagonalizable. (6 pts.)

(d) If A is a 2×2 matrix over R and $A^2 = I$, then A is diagonizable.
This is true. If A is multiplication by a scalar, then A is diagonal. If not, there exists a vector v such that Av is not a multiple of v, and then $A^2 v = v$. Then the matrix for L_A in the basis (v, Av) is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the characteristic polynomial of A is $\lambda^2 - 1$. Since this splits and has distinct roots, A is diagonlizable. (6 pts.)

(e) If A and B are two $n \times n$ matrices with the same set of eigenvalues, then A and B are similar.
This is false. For example $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has the same eigenvalues as the zero matrix, but it is not diagonlizable. (6 pts)