Mathematics 110 Professor A. Ogus

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## Linear Algebra Midterm Exam

Write clearly, with complete sentences, explaining your work. You will be graded on clarity, style, and brevity. If you add false statements to a correct argument, you will lose points.

1. Answer the following questions using precise mathematical language.

   (a) What is the definition of a *linearly independent* subset of a vector space?
   A subset $S$ is linearly independent if and only if whenever $(x_1, \cdots x_n)$ is a finite sequence of distinct element of $S$ and $(a_1, \cdots a_n)$ is a sequence in $F$ such that $\sum a_i x_i = 0$, each $a_i = 0$.

   (b) What is the definition of the *dimension* of a vector space? What is the key fact that is needed for this definition to make sense?
   The dimension of a vector space is the number of elements in any basis for the space. For this to make sense, we need to know that every vector space has a basis, and especially that any two bases for the same space have the same number of elements.

   (c) What is the definition of a *linear transformation*?
   A linear transformation is a function $T$ from a vector space $V$ over $F$ to a vector space $W$ over $F$ such that for any $v, v'$ in $V$ and any $c \in F$, $T(cv + v') = cT(v) + T(v')$.

   (d) What is the definition of the *dual* of a vector space?
   The dual $V^*$ of a vector space $V$ is the space $L(V, F)$ of linear transformations $V \to F$, viewed as a linear subspace of the space of all functions $V \to F$.

   (e) What is the definition of the *transpose* of a linear transformation between vector spaces?
   Let $T\colon V \to W$ be a linear transformation. Then the transpose $T^*$ of $T$ is the map $W^* \to V^*$ sending $g$ to $g \circ T$.

2. Let $V$ and $W$ be vector spaces, let $T: V \to W$ be a linear transformation, and suppose that $(v_1, \ldots, v_n)$ is an ordered basis for $V$ such that $(v_{k+1}, \ldots, v_n)$ is an ordered basis for the null space of $T$. Prove that $\{T(v_1), \ldots T(v_k)\}$ is a linearly independent subset of $W$.
Suppose that $(a_1, \cdots a_k)$ is a sequence of elements of $F$ such that $\sum a_i T(v_i) = 0$. Then $T(\sum_1^k a_i v_i) = 0$, so $\sum a_i v_i$ belongs to the null space of $T$. Since $(v_{k+1} \cdots, v_n)$ is a basis for the null space, $\sum_1^k a_i v_i = \sum_{k+1}^n a_i v_k$ for some elements $a_i \in F$, $i = k+1, \cdots, n$. Since the sequence $(v_1, \cdots v_n)$ is an ordered basis for $V$, all $a_i = 0$.

3. Let $V$ be the vector space of functions from the reals to the reals. If $f \in V$, let $T(f)(x) := f(x-1)$.

   (a) Show that $T$ defines a linear transformation from $V$ to $V$. Be sure to use the definition of the vector space structure of $V$.
   Let $f$ and $g$ be elements of $V$ and let $c$ be an element of $R$. We have to show that $T(cf + g) = cT(f) + T(g)$, and this amounts to proving that $T(cf + g)(x) = (cT(f) + T(g))(x)$ for all $x \in R$. By the definition of the vector space structure on $V$, this latter is $cT(f)(x) + T(g)(x)$. Thus we must show that $(cf + g)(x - 1) = cf(x-1) + g(x-1)$, which is true by the definition of addition and scalara multiplication.

   (b) Let $W$ denote the space of polynomials of degree less than or equal to 2, regarded as a subpspace of $V$. Using the fact that $T(f)$ is again in $W$, compute the matrix representing $T$ with respect to the basis $\{1, x, x^2\}$ for $W$. We compute

$$T(1) = 1, T(x) = x - 1, T(x^2) = (x-1)^2 = x^2 - 2x - 1.$$

   Thus the matrix for $T$ is $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

4. Say whether each of the following is true or false. In each case, explain your answer. In particular, if you claim that statement is true, give a brief proof, and if you claim it is false, give a counterexample.

(a) No linear transformation from $R^3$ to $R^2$ is surjective.
This if false. For examle, the projection sending $(a, b, c)$ to $(a, b)$ is surjective.

(b) No linear transformation from $R^3$ to $R^2$ is injective.
This is true, because the dimension of $R^3$ is larger than the dimension of $R^2$. The nullity rank theorem implies, in fact, that the null spaced of any such transformation has dimension at least one.

(c) Every vector space of dimension at least two over $R$ is infinite.
This is true. Any such space contains a nonzero vector $v$, and for any two distinct real numbers $a$ and $b$, $av \neq bv$. Since $R$ is infinite, so is $V$.

(d) Every vector space of dimension at least two over $R$ contains infinitely many distinct linear subspaces.
This is true. Since the dimension is at least two, there exists a linearly independent pair of vectors $(v, w)$. Now I claim that if $a$ and $b$ are distinct real numbers then the span of $v + aw$ and the span of $v + bw$ are distinct linearly subspaces. Otherwise, there would exist a $c$ such that $v + bw = c(v + aw)$. But the linear independence shows that $c = 1$ and $a = b$, a contradiction.

(e) Every finite dimensional vector space is isomorphic to its dual.
This is true, since a vector space has the same dimension as its dual.