1.5.5

Suppose the set is linearly dependent. Then $\exists\ a_0, a_1, \ldots, a_n \in F,$ not all zero, such that $a_0 + a_1\text{x} + \ldots + a_n\text{x}^n = 0\ \forall\ \text{x}\in\text{F}$. However, this would mean that there is a polynomial other than the zero polynomial that has infinitely many roots. This cannot be, since the maximum number of roots a polynomial of degree n can have is n. Thus $a_0 + a_1\text{x} + \ldots + a_n\text{x}^n$ will be 0 if and only if $a_0 = a_1 = \ldots = a_n = 0$. Hence our set is linearly independent.

1.6.12

Since there are three vectors in the basis for V, dim V = 3. By **Corollary 2b**, any linearly independent subset of V that contains exactly dim V = n elements is a basis for V. Thus, if the set $\{u+v+w, v+w, w\}$ is linearly independent, it is a basis for V, since it contains 3 elements.

We know that, since $\{u, v, w\}$ form a basis for V, they are linearly independent, and so $a_1\text{u} + a_2\text{v} + a_3\text{w} = 0$ iff $a_1 = a_2 = a_3 = 0$. Check when c_1 (u+v+w) + c_2 (v+w) + c_3 w = c_1u + $(c_1 + c_2)$v + $(c_1+c_2+c_3)$w = 0. From the independence of $\{u, v, w\}$, we know that

$$c_1=0$$
$$c_1 + c_2=0$$
$$c_1+c_2+c_3=0$$

Substituting $c_1=0$ into $c_1 + c_2=0$ we get $c_2=0$, and substituting $c_1 + c_2=0$ into $c_1+c_2+c_3=0$ we get $c_3=0$. So $c_1=c_2=c_3=0$, implying that $\{u+v+w, v+w, w\}$ is linearly independent. Thus it is a basis for V.

1.6.23

a) dim W_1 = dim W_2 iff $v = a_1 v_1 + a_2v_2 + \ldots + a_nv_n$ for some a_i in the field F:

($\Rightarrow$) Suppose dim W_1 = dim W_2 = k. WOLOG, let $\{v_1, \ldots , v_k\}$ then be the largest linearly independent subset of $\{v_1, \ldots , v_n\}$. Now suppose there are no such a_i such that $v = a_1 v_1 + a_2v_2 + \ldots + a_nv_n$. Then there are also no a_i such that $v = a_1 v_1 + a_2v_2 + \ldots + a_kv_k$. So $\{v_1,\ldots,v_k,v\}$ is the largest linearly independent subset of $\{v_1,\ldots,v_n,v\}$, which spans W_2. So it is also a basis for W_2, and so dim W_2 = k+1 $\neq$ k = dim W_1. This is a contradiction, so $v = a_1 v_1 + a_2v_2 + \ldots + a_nv_n$ for some a_i in the field F.

($\Leftarrow$) Suppose $v = a_1 v_1 + a_2v_2 + \ldots + a_nv_n$ for some a_i in the field F. Then v is in span $\{v_1,\ldots,v_n\}$, and so W_2 = span $\{v_1,\ldots,v_n,v\}$ = span $\{v_1,\ldots,v_n\}$ = W_1. Hence dim W_1 = dim W_2.

b) In the case that dim $\text{W}_1 \neq$ dim W_2, we must have dim W_1 + 1 = dim W_2. dim $\text{W}_1 \neq$ dim W_2 implies there is no such a_i such that $v = a_1 v_1 + a_2v_2 + \ldots + a_nv_n$. Thus, if dim $\text{W}_1 = k$, dim $\text{W}_2 = k + 1$ (see a).