2.1.22

Proof for $\mathbf{F}^n \to \mathbf{F}^m$: (we can extend this to all other cases in the problem easily)

Since $\mathbf{F}^n$ is a vector space of dimension n, define the elementary basis on it, $\{e_1, \ldots, e_n\}$. Then any vector $(a_1, \ldots, a_n)$ can be expressed as a linear combination of our basis, namely $a_1e_1 + \ldots + a_ne_n$, where $a_i \in \mathbf{F}$. Then

$$T(a_1, \ldots, a_n) = a_1\, T(e_1) + \ldots + a_n\, T(e_n),$$

since T is linear. $T(e_i)$ are vectors in $\mathbf{F}^m$, so we have that $\exists$ vectors in $\mathbf{F}^m$ such that the above holds for any vector $(a_1, \ldots, a_n)$. To extend this to the case $\mathbf{R}^3 \to \mathbf{R}$, we simply note that $\mathbf{R}$ a case in which m=1, so the vectors in it are simply reals. In the case $\mathbf{F}^n \to \mathbf{F}$, we have m=1, and the vectors in $\mathbf{F}$ are just elements of the field.

2.1.28

<u>{0} is T-invariant:</u>
Since T is linear, $T(0) = 0$, so $\{0\}$ is mapped to itself and is thus T-invariant.
<u>V is T-invariant:</u>
Since $T: V \to V$, $T(v) \in V\ \forall v \in V$. Thus V is T-invariant.
<u>R(T) is T-invariant:</u>
Note that $R(T) \subseteq V$, and note that $T(v) \in R(T)\ \forall v \in V$. Thus, in particular, $T(v) \in R(T)\ \forall v \in R(T)$. So $T(R(T)) \subseteq R(T)$, and so R(T) is T-invariant.
<u>N(T) is T-invariant:</u>
By definition, $T(v) = 0\ \ \forall v \in N(T)$. Since T is linear, $T(0) = 0 \in N(T)$, so $T(v) \subseteq N(T)\ \forall v \in N(T)$. Thus N(T) is T-invariant.