Row Equivalence of matrices

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0.1 Row equivalence

Let F be a field and let m and n be positive integers. Two m by n matrices are said to be *row equivalent* if there is an invertible matrix S such that $B = SA$. (Check that this is indeed an equivalence relation.) The textbook shows that any two row equivalent matrices have the same null space. In fact the converse is also true, so that we have the following theorem:

Theorem: If A and B are two m by n matrices, then the following conditions are equivalent:

1. There exists an invertible matrix S such that $B = SA$

2. The matrices A and B have the same nullspace.

This theorem can be translated into a statement about linear transformations by considering the linear transformations L_A, L_B, and L_S associated with the matrices above. It seems more revealing to treat this more abstract looking version.

Theorem Let V and W be finite dimensional vector spaces, and let T and T' be linear transformations from V to W. Then the following are equivalent:

1. There exists an isomorphism $S: T \to T'$ such that $T' = S \circ T$.

2. The transformations T and T' have the same kernel.

Proof: The proof that (1) implies (2) is relatively easy (and is in the book) and we omit it. Let us prove that (2) implies (1). Let $K := Ker(T) = Ker(T')$, a linear subspace of V. By the rank-nulity theorem, the dimension k of K is $n-r$, where r is the rank of T and hence also the rank of T'. Choose

a basis for K, and prolong it to a basis for V. Now choose an ordering $(v_1, v_2, \cdots v_n)$ for this basis so that the last k vectors, that is, the vectors $(v_{r+1}, \ldots v_n)$, are the chosen basis for K. For $1 \leq i \leq r$, let $w_i := T(v_i)$ and let $w_i' := T(v_i')$.

Lemma: Each of the sequences $(w_1, \ldots w_r)$ is $(w_1', \ldots w_r')$ is linearly independent.

This fact was proved during the proof of the rank nullity theorem, but let us repeat the proof. We just write it for the sequence $(w_1, \ldots, w_r)$. Suppose that $a_1 w_1 + \cdots a_r w_r = 0$. Then $T(a_1 v_1 + \cdots a_r v_r) = a_1 T(v_1) + \cdots a_r T(v_r) = a_1 w_1 + \cdots a_r w_r = 0$. This means that $a_1 v_1 + \cdots a_r v_r \in Ker(T) = K$. Since $(v_{r+1}, \ldots v_n)$ spans K, we can write $a_1 v_1 + \cdots a_r v_r = b_{r+1} v_{r+1} + \cdots b_n v_n$ for a suitable choice of b_j's. But since the sequence $(v_1, \ldots v_n)$ is linearly independent, all the a_i's and b_j's are zero. This completes the proof of the lemma.

Now extend $(w_1, \ldots w_r)$ to a basis $(w_1, \ldots w_m)$ for W and extend $(w_1', \ldots w_r')$ to a basis $(w_1', \ldots w_m')$ to a basis for W. By Theorem 2.6 of the text, there is a unique linear transformation S such that $S(w_i) = w_i'$ for all i. In fact, S is invertible because $(w_1', \ldots w_m')$ is also a basis for W. Thus to prove the theorem it will be enough to show that $T' = S \circ T$. Since both T' and $S \circ T$ are linear and $(v_1, \ldots v_n)$ is a basis for V, it will suffice to prove that $T'(v_i) = S \circ T(v_i)$ for each i, If $1 \leq i \leq r$, we find $T'(v_i) = w_i' = S(w_i) = S(T(w_i)) = T \circ S(w_i)$, as required. If $i > r$, $v_i \in K = Ker(T) = Ker(T')$, and $S \circ T(v_i) = S(T(v_i)) = S(0) = 0 = T(v_i)$. This completes the proof.