Math 110, Professor Ogus, Homework due 3/14

(written by Janak Ramakrishnan)

5.1.1. (a) F, (b) T, (c) T, (d) F, (e) F, (f) F, (g) F, (h) T, (i) T, (j) F, (k) F.

5.1.2a.

$$\begin{aligned} T(1,2) &= (10-12, 17-20) = (-2,-3) = -(2,3) \\ T(2,3) &= (20-18, 34-30) = (2,4) = 2(1,2) \\ [T]_\beta &= \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}. \end{aligned}$$

β is not a basis of eigenvectors, since each vector is sent to the other one, and not sent to a scalar multiple of itself.

5.1.2b.

$$\begin{aligned} T(3+4x) &= (18-24) + (36-44)x = -6-8x = -2(3+4x) \\ T(2+3x) &= (12-18) + (24-33)x = -6-9x = -3(2+3x) \\ [T]_\beta &= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}. \end{aligned}$$

β is a basis of eigenvectors, since each vector is sent to a scalar multiple of itself.

5.1.3b. $\operatorname{Det}(A-\lambda I) = 6 - 11\lambda + 6\lambda^2 - \lambda^3$. By inspection, $1, 2, 3$ are roots, and thus the eigenvalues of A are $1, 2, 3$. $A(x,y,z) = (-2y-3z, -x+y-z, 2x+2y+5z)$. We find the eigenvector for each eigenvalue:

$$\begin{aligned} (-2y-3z, -x+y-z, 2x+2y+5z) &= (x,y,z) \\ -2y-3z &= x \\ -x+y-z &= y \\ x &= -z \\ -2y-2z &= 0 \\ y &= -z. \end{aligned}$$

Thus, $(1, 1, -1)$ is an eigenvector with eigenvalue 1.

$$\begin{aligned}(-2y - 3z, -x + y - z, 2x + 2y + 5z) &= (2x, 2y, 2z)\\ -x + y - z &= 2y\\ 2x + 2y + 5z &= 2z\\ -z &= x + y\\ -2z + 5z &= 2z\\ z &= 0\\ y &= -x.\end{aligned}$$

Thus, $(1, -1, 0)$ is an eigenvector with eigenvalue 2.

$$\begin{aligned}(-2y - 3z, -x + y - z, 2x + 2y + 5z) &= (3x, 3y, 3z)\\ -2y - 3z &= 3x\\ -x + y - z &= 3y\\ -x - z &= 2y\\ -2y + 3y &= 0\\ y &= 0\\ z &= -x.\end{aligned}$$

Thus, $(1, 0, -1)$ is an eigenvector with eigenvalue 3.

5.1.3c. $\mathrm{Det}(A - \lambda I) = -1 + \lambda^2$. By inspection, 1 and -1 are roots. $A(x, y) = (ix + y, 2x - iy)$. We find the eigenvector for each eigenvalue:

$$\begin{aligned}(ix + y, 2x - iy) &= (x, y)\\ ix + y &= x\\ y &= (1 - i)x.\end{aligned}$$

Thus, $(1, 1 - i)$ is an eigenvector with eigenvalue 1.

$$\begin{aligned}(ix + y, 2x - iy) &= (-x, -y)\\ ix + y &= -x\\ y &= -(1 + i)x.\end{aligned}$$

Thus, $(1, -1 - i)$ is an eigenvector with eigenvalue -1.

5.1.4e. We write $f(x)$ as $a + bx + cx^2$. Then $T(f(x)) = x(b + 2cx) + (a + 2b + 4c)x + (a + 3b + 9c) = (a + 3b + 9c) + (a + 3b + 4c)x + 2cx^2$. In the basis we have chosen, T's matrix form is then

$$\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2. \end{pmatrix}$$

Then its characteristic polynomial is $-8\lambda + 6\lambda^2 - \lambda^3$, so its eigenvalues are $0, 2, 4$. Then we have, for each:

$$\begin{aligned} a + 3b + 4c &= 0 \\ 2c &= 0 \\ a &= -3b. \end{aligned}$$

Thus, $-3 + x$ is an eigenvector.

$$\begin{aligned} a + 3b + 9c &= 2a \\ a + 3b + 4c &= 2b \\ a &= 3b + 9c \\ 6b + 13c &= 2b \\ -4b &= 13c. \end{aligned}$$

Thus, $3 + 13x - 4x^2$ is an eigenvector.

$$a + 3b + 4c = 4b2c = 4cc = 0a = b.$$

Thus, $1 + x$ is an eigenvector, so $\beta = \{1 + x, -3 + x, 3 + 13x - 4x^2\}$.

5.1.4h. Note that V is 4-dimensional, and by inspection, it is easy to see that $\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$ have eigenvalue 1, and that $\begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}$ has eigenvalue -1. Thus, the eigenvalues are 1 (with multiplicity 3) and -1, and these matrices form a basis such that T is a diagonal matrix.

5.1.8. (a) If 0 is an eigenvalue of T, then there is a vector v such that $Tv = 0v$. But then $Tv = 0$, and so T has a non-trivial null space, and therefore is not invertible. If T is not invertible, then $Tv = 0$ for some v, and so $Tv = 0v$, and so v is an eigenvector.

(b) Let λ be an eigenvalue of T, with an eigenvector v. Then $T(v) = \lambda v$, so $T^{-1}(\lambda v) = v$, and thus, by linearity, $T^{-1}(v) = \lambda^{-1}v$, and so λ^{-1} is an eigenvalue of T^{-1}.

(c) The precise statements of a and b are true for matrices, considering them as linear transforms (via the L_A map).

5.1.14. The characteristic polynomial of A^t is $\text{Det}(A^t - \lambda I) = \text{Det}(A^t - (\lambda I)^t) = \text{Det}((A - \lambda I)^t) = \text{Det}(A - \lambda I)$ which is the characteristic polynomial of A, since λI is diagonal, the sum of transposes is the transpose of the sum, and the transpose does not alter the determinant

5.1.15a. By induction. Clear for $m = 1$. If $m = k+1$, $T^m(x) = T(T^k(x)) = T(\lambda^k x) = \lambda^k T(x) = \lambda^{k+1}x = \lambda^m x$.

5.1.16. (a) Let A and B be similar matrices, with $A = QBQ^{-1}$. Then $\text{Tr}(A) = \text{Tr}(Q^{-1}BQ) = \text{Tr}(QQ^{-1}B) = \text{Tr}(B)$.

(b) The trace of a linear operator should be defined as the sum of its eigenvalues (with multiplicity). From problem 12, we know that this definition is independent of similarity transforms. It is easy to see that if a linear operator T is diagonalizable, then this definition agrees with the usual definition of trace. Later in the book, we will see that this definition agrees with the usual one for all linear operators.

5.1.22a. Let $g(t) = a_0 + a_1 t + \ldots a_n t^n$. Then $g(T)(x) = (a_0 + a_1 T + \ldots a_n T^n)(x) = a_0 x + a_1 T(x) + \ldots + a_n T^n(x) = a_0 x + a_1 \lambda x + \ldots + a_n \lambda^n x = (a_0 + a_1\lambda + \ldots + a_n \lambda^n)x = g(\lambda)x$.

5.1.23. We show that $f(T)$ and T_0 agree on a basis, which is sufficient, since both are linear operators. Let β be an eigenbasis for T, which exists since T is diagonalizable. Let v be an arbitrary element of β, with eigenvalue λ. Then $f(T)(v) = f(\lambda)(v)$, by exercise 22a, and then, since the eigenvalues of T are the roots of the characteristic polynomial, $f(\lambda) = 0$, and so $f(T)(v) = 0$. Thus, $f(T)$ is 0 on every element of β, and is thus equal to T_0.

5.2.1. (a) F, (b) F, (c) F, (d) T, (e) T, (f) F, (g) T, (h) T, (i) F.

5.2.2a. A has characteristic polynomial $(\lambda - 1)^2$. The equation $A(x, y) = (x, y)$ is equivalent to $x + 2y = x$ and $y = y$. The first equation yields $y = 0$. Thus, $(1, 0)$ spans the eigenspace E_1, and so A does not have an eigenbasis, and is not diagonalizable.

5.2.2c. A has characteristic polynomial $\lambda^2 - 3\lambda - 10$, with roots -2 and 5. Thus it must have an eigenbasis. Eigenvectors are obtained by solving: $A(x, y) = -2(x, y)$, so $x + 4y = -2x$, and $y = -3x/4$, so $(4, -3)$ is an eigenvector; $A(x, y) = 5(x, y)$, so $x + 4y = 5x$, so $x = y$, and $(1, 1)$ is an eigenvector. Thus, $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$, and $Q = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$.

5.2.2e. A has characteristic polynomial $1 - \lambda + \lambda^2 - \lambda^3$. By inspection, it has 1 as a root. Factoring that out, we get $1 + \lambda^2$. Since this does not split over $\mathbb{R}$, A is not diagonalizable over $\mathbb{R}$, although it is over $\mathbb{C}$.

5.2.3a. Writing $f(x)$ as $a + bx + cx^2 + dx^3$, $T(f(x)) = b + 2cx + 3dx^2 + 2c + 6dx = (b + 2c) + (2c + 6d)x + 3dx^2$. If a polynomial, f, has a non-zero x^3 term, then $T(f)$ has a non-zero x^2 term, but no x^3 term, so f cannot be an eigenvector. Thus, any eigenvector of T has no x^3 term, and T cannot have an eigenbasis.

5.2.3f. It is clear that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ have eigenvalue 1, and that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has eigenvalue -1. Thus, these vectors give us β.

5.2.7. We know from 2c that we can write $A = QDQ^{-1}$, where $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$, and $Q = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$. Then $A^n = (QDQ^{-1})^n = QD^nQ^{-1} = Q\begin{pmatrix} (-2)^n & 0 \\ 0 & 5^n \end{pmatrix}Q^{-1}$, which can also be written as

$$\begin{pmatrix} (-2)^n - 5^n/3 & 4(-2)^n + 5^n \\ -3(-2)^n/4 - 5^n/3 & -3(-2)^n + 5^n \end{pmatrix}.$$

5.2.12. The argument of 5.1.8 shows that every eigenvector of T with eigenvalue λ is an eigenvector of T^{-1} with eigenvalue λ^{-1}. This shows that the eigenspaces in question are equal. For part (b), let $\{v_1, \ldots, v_n\}$ be an eigenbasis for T. Then the v_i's are still eigenvectors of T^{-1}, by part (a), and still linearly independent, so they are an eigenbasis for T^{-1}.

5.2.13. (a) If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then E_1 is spanned by $(1, 0)$ but E_1' is spanned by $(0, 1)$.

(b) $\dim(E_\lambda') = \dim(N(A^t - \lambda I)) = \dim(N(A^t - (\lambda I)^t)) = \dim(N((A - \lambda I)^t)) = \dim(V) - \text{Rank}((A - \lambda I)^t) = \dim(V) - \text{Rank}(A - \lambda I) = \dim(N(A - \lambda I)) = \dim(E_\lambda)$.

(c) If A is diagonalizable, then its characteristic polynomial splits, and each eigenspace has full dimension. Since A^t has the same characteristic polynomial, its characteristic polynomial also splits, and since each of its eigenspaces has the same dimension as the corresponding eigenspaces for A, its eigenspaces also have full dimension, and so it splits.