Math 110, Professor Ogus, Homework due 3/7

(written by Janak Ramakrishnan)

3.3.1. (a) F, (b) T, (c) F, (d) F, (e) F, (f) F, (g) T

3.3.3b. $(5-2i)(7i)-(6+4i)(-3+i) = 35i+14+18+12i-6i+4 = 36+41i$.

3.3.7. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A^t) = ad - cb = ad - bc = \det(A)$.

4.2.1. (a) F, (b) T, (c) T, (d) T, (e) F, (f) F, (g) F, (h) T.

4.2.4.

$$\det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} + \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} =$$

$$\det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ b_1 & b_2 & b_3 \end{pmatrix} + \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1 & a_2 & a_3 \end{pmatrix} +$$

$$\det\begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} + \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} =$$

$$0 + \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + \det\begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + 0 + \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} =$$

$$0 + (-1)^2 \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} + 0 + (-1)^2 \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 2\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Thus, $k = 2$.

4.2.10.

$$\det\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix} = (-1)(-1)(2+i)(1-i) + 3i(1-i) - 2i(-i) = 4+2i$$

4.2.25. $kA = kIA = (kI)A$. Thus, $\det(kA) = \det((kI)A) = \det(kI)\det(A)$. Since kI is upper-triangular, $\det(kI)$ is the product of the diagonal entries, which is k^n, so $\det(kA) = k^n \det(A)$.

4.3.1. (a) F, (b) T, (c) F, (d) T, (e) F, (f) T, (g) F, (h) F.

4.3.5.

$$x_1 = \det\begin{pmatrix} -4 & -1 & 4 \\ 8 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix} = -20, \qquad x_2 = \det\begin{pmatrix} 1 & -4 & 4 \\ -8 & 8 & 1 \\ 2 & 0 & 1 \end{pmatrix} = -48, \qquad x_3 = \det\begin{pmatrix} 1 & -1 & -4 \\ -8 & 3 & 8 \\ 2 & -1 & 0 \end{pmatrix} = -8$$

4.3.21. By performing elementary row operations on C, we can reduce it to the form I or to a matrix with a zero row. Let the elementary row operations used be $E_1, \ldots, E_k$. Let A be $m \times m$. For each elementary matrix E_i, let E_i' be the $n \times n$ elementary matrix corresponding to its action on C in M, so that if E_i switches rows j and l, then E_i' switches rows $j+m$ and $l+m$. It is easy to see that $\det(E_i') = \det(E_i)$. Then $E_1' \cdots E_k' M$ is a matrix which is either in the form $\begin{pmatrix} A & B \\ O & I \end{pmatrix}$ or $\begin{pmatrix} A & B \\ O & P \end{pmatrix}$, where P has a zero row. In the second case, $\det(E_1' \cdots E_k' M)$ is 0, since the matrix has a zero row, and so $\det(M) = 0$, and since $\det(C) = 0$ (as P, C's RREF, had a zero row), $\det(M) = \det(A)\det(C)$. In the first, by exercise 20, $\det(E_1' \cdots E_k' M) = \det(A)$. Thus, $\det(M) = \det(A)/(\det(E_1' \cdots E_k')) = \det(A)/(\det(E_1') \cdots \det(E_k')) = \det(A)/(\det(E_1) \cdots \det(E_k))$. Now, since in this case C is invertible, we know that $\det(C) = (\det(E_1) \cdots \det(E_k))^{-1}$, and thus $\det(M) = \det(A)\det(C)$.

4.4.1. (a) T, (b) T, (c) T, (d) F, (e) F, (f) T, (g) T, (h) F, (i) T, (j) T, (k) T.

4.4.6. See 4.3.21.

4.5.3. Not 3-linear:

$$\delta \begin{pmatrix} 2(0 & 0 & 0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = k \neq 2\delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2k.$$

4.5.6. Not 3-linear:

$$\delta \begin{pmatrix} 1 & 0 & 0 \\ 2(0 & 0 & 0) \\ 0 & 0 & 0 \end{pmatrix} = 1 \neq 2\delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2.$$

4.5.16. Given any matrix A, we can transform it into a matrix B in RREF. By a previous exercise, we need only use operations of type 2 and 3 (scaling rows and adding a scalar multiple of one row to another). By Corollary 1, an operation of type 3 does not alter δ, and by n-linearity, an operation of type 2 scales δ by c, where c was the scalar used to multiply the row. Thus, let $B = E_1 \cdots E_k A$, where each E_i is an elementary matrix of type 2 or 3. Let $c_i = 1$ if E_i is of type 3, and let c_i be the scalar that E_i multiplied a row by, if E_i is of type 2. Then we know that $\delta(B) = c_1 \cdots c_k \delta(A)$. But either B has a zero row, in which case it is easy to see, by n-linearity, that $\delta(B) = 0$, or $B = I$. Note that $\det(B) = c_1 \cdots c_k \det(A)$. If $\det(B) = 0$, then $\delta(B) = 0$, so $\delta(A) = 0$ (and $\det(A) = 0$), and so $\delta(A) = \delta(I)\det(A)$. If $\det(B) = 1$, then $\det(A) = c_1 \cdots c_k$, and so $\det(A) = \delta(B)\det(A) = \delta(I)\det(A)$. Thus, we are done with $k = \delta(I)$.