Math 110, Professor Ogus, Homework due 2/28

(written by Janak Ramakrishnan)

3.3.1. (a) F, (b) F, (c) T, (d) F, (e) F, (f) F, (g) T, (h) F.

3.3.2. d. Adding the first two equations together, we note that $3x_1 = 0$, so $x_1 = 0$. Then from the second equation, $x_2 = x_3$. Thus, a basis is $(0,1,1)$ and the dimension is 1.

g. These two equations are clearly linearly independent (one has x_1 and the other does not). Subtracting the second equation from the first, we get $x_1 + x_2 + 2x_3 = 0$. Then the dimension is 2, since we have 2 equations and 4 unknowns, and $(2,0,-1,-1)$ and $(0,2,-1,-3)$ are a basis.

3.3.3. d. Repeating the above argument, we see that for one solution, $x_1 = 2$, and so $x_3 - x_2 = -1$, $2(x_2 - x_3) = 2$, so $x_2 - x_3 = 1$, and thus $x_3 = 0$, $x_2 = 1$. Therefore, the set of all solutions is $\{(x_1, x_2, x_3) \in F^3 \mid x_1 = 2, x_2 = 1 + t, x_3 = t, t \in F\}$, where F is our base field.

g. Again subtracting the second equation from the first, we have $x_1 + x_2 + 2x_3 = 0$. Thus, $(2,0,-1,0)$ is a solution. Therefore, the set of all solutions is $\{(x_1, x_2, x_3, x_4) \in F^4 \mid x_1 = 2t+2, x_2 = 2s, x_3 = -1 - t - s, x_4 = -t - 3s, t, s \in F\}$.

3.3.6. We have the equations $a + b = 1$ and $2a - c = 11$. The easiest way to solve these is to make $a = 0$, $b = 1$, and $c = -11$. The set of all solutions to $T(\vec{v}) = \vec{0}$ is spanned by $(1,-1,2)$. Thus $T^{-1}(1,11) = \{(a,b,c) \mid a = t, b = 1 - t, c = -11 + 2t, t \in \mathbb{R}\}$.

3.3.8. Note that $R(T)$ has dimension 2, since $(1,0,1)$ and $(1,1,0)$ are in it, but if $(x,y,z) \in R(T)$, then $x = y + z$, so in particular $R(T) \neq \mathbb{R}^3$. Since $\{(x,y,z) \in \mathbb{R}^3 \mid x = y + z\}$ has dimension 2 and contains $R(T)$, it must be $R(T)$, so we need only check this equation. $1 \neq 3 + 2$, so $(1,3,2) \notin R(T)$, but $2 = 1 + 1$, so $(2,1,1) \in R(T)$.

3.4.1. (a) F, (b) T, (c) T, (d) T, (e) F, (f) T, (g) T.

3.4.2d. We form the augmented matrix and solve:

$$\left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 2 & -1 & 6 & 6 & -2 \\ -2 & 1 & -4 & -3 & 0 \\ 3 & -2 & 9 & 10 & -5 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & -1 & -8 & 3 & -14 \\ 0 & 1 & 15 & 1 & 16 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 5 & 1 & 4 \end{array}\right) \to$$

$$\left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 1.5 & -1 \\ 0 & 0 & 5 & 1 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 3/2 & -1 \\ 0 & 0 & 0 & -13/2 & 9 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 3/2 & -1 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right) \to$$

$$\left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 1 & 0 & 14/13 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 0 & 0 & 16/13 \\ 0 & 0 & 1 & 0 & 14/13 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 7/13 \\ 0 & 1 & 0 & 0 & 16/13 \\ 0 & 0 & 1 & 0 & 14/13 \\ 0 & 0 & 0 & 1 & -18/13 \end{array}\right)$$

Thus, the solution is $(7/13, 16/13, 14/13, -18/13)$.

3.4.5. Let the rows of the RREF of A be v_1, v_2, v_3. Since each row of A is a linear combination of the rows in the RREF of A, we know that the first row of A must equal $v_1 + v_3$. Thus, the third coordinate is 2. The second row of A must be $-v_1 - v_2 - 2v_3$, so the third coordinate is $-2 + 5 = 3$. Finally, the third row of A must be $3v_1 + v_2$, so the third coordinate is 6.

3.4.13. Let $v_1 = (1, 0, 1, 1, 1, 0)$ and $v_2 = (0, 2, 1, 1, 0, 0)$, so $S = \{v_1, v_2\}$. For part a, it is clear that v_1 and v_2 are linearly independent (v_1 has a nonzero first coordinate). Since $1 - 0 + 0 \cdot 1 + 2 \cdot 1 - 3 \cdot 1 + 0 = 0$ and $2 \cdot 1 - 0 - 1 + 3 \cdot 1 - 4 \cdot 1 + 4 \cdot 0 = 0$, $v_1 \in V$, and likewise, $2 \cdot 0 - 2 + 0 \cdot 1 + 2 \cdot 1 - 3 \cdot 0 + 0 = 0$ and $2 \cdot 0 - 2 + -1 + 3 \cdot 1 - 4 \cdot 0 + 4 \cdot 0 = 0$ show that $v_2 \in V$. It is easy to verify that V is 4-dimensional. The vectors $v_3 = (0, 0, -5, 1, 0, -2)$ and $v_4 = (0, 0, 1, 3, 2, 0)$ complete S to a basis – it is easy to check that they lie in V, so we need only check that $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set. But v_3 is clearly independent from $\{v_1, v_2\}$, since its last coordinate is nonzero, and both v_1 and v_2 have zero last coordinate. v_4 is independent from v_1, v_2, v_3 because any linear combination of v_1, v_2, v_3 with nonzero coefficient for v_1 must have nonzero first coordinate, and likewise for v_2. Thus, we need only consider multiples of v_3, but v_3 and v_4 are easily linearly independent.

3.4.15. Let B and B' be two matrices in RREF, with B,B' equal to A in RREF. We know from the theorem that B and B' are $m \times n$, with r nonzero rows, where $r = \text{Rank}(A)$, or else there is no way for B' to be the RREF of A. We will show $B = B'$.

First we show that for each $i \le n$, if we take the first j such that either $b_j = e_i$ or $b'_j = e_i$, then $b_j = b'_j = e_i$. We show this by contradiction, so assume that it is not true for some i. We can choose the least such i. WLOG, we can assume that $b_j = e_i$ – else we can switch B and B'. For B' to be in RREF, since e_i is not in the first j columns of B', b'_j cannot have any nonzero component of $e_i, \ldots, e_r$. Then it is easy to see that b_j is a linear combination of $e_1, \ldots, e_{i-1}$, say $d'_1 e_1 + \ldots + d'_{i-1} e_{i-1}$. If we let j_k (for $k < i$) be the first column at which $b_{j_k} = b'_{j_k} = e_k$, then by Theorem 3.16(d), this means that $a_j = d'_1 a_{j_1} + \ldots d'_{j_{i-1}} a_{j_{i-1}}$. But now consider B and Theorem 3.16(c): we can let $j_i = j$, and so we know that a_j is linearly independent from $a_{j_1}, \ldots, a_{j_{i-1}}$, which is a contradiction. Thus, there is no such i – for each i and the first j such that $b_j = e_j$, $b'_j = e_j$ as well.

Thus, we can choose $j_1, \ldots, j_r$ as in Theorem 3.16 to be the same for B and B'. Now we note that the converse of Theorem 3.16(d) is also true, since there is only one possible representation of each column of A in terms of $a_{j_1}, \ldots, a_{j_r}$, as they are linearly independent, by Theorem 3.16(c). Thus, column k of A determines column k of B and of B', showing that $B = B'$.