Assignment 5

MATH110

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Section 2.3: 1, 3, 8, 9, 11, 15; Section 2.4: 1, 2a,c,f , 5, 10; Section 2.5: 1, 2ac, 6, 10, 11

3.1 (a) False.
(b) True.
(c) False.
(d) True.
(e) False.
(f) False.
(g) True.
(h) False.
(i) True.
(j) True.

3.3 $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$, $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$,

3.8 **Theorem 2.10** Let V be a vector space. Let $T, U_1, U_2 \in \mathcal{L}(V)$. Then

(a) $T(U_1 + U_2) = TU_1 + TU_2$
Proof: $T(U_1 + U_2)(v) = T(U_1(v) + U_2(v)) = TU_1(v) + TU_2(v)$.

(b) $T(U_1U_2) = (TU_1)U_2$
Proof: $T(U_1U_2)(v) = T((U_1U_2)(v)) = T(U_1(U_2(v))) = (TU_1)(U_2(v)) = (TU_1)(U_2)(v)$.

(c) $TI = IT = T$
Proof: $TI(v) = T(I(v)) = T(v) = I(T(v)) = IT(v)$

(d) $a(U_1U_2) = (aU_1)(U_2) = U_1(aU_2)$ for all scalars a.
Proof: $a(U_1U_2)(v) = a(U_1(U_2(v))) = U_1(aU_2(v))$.

General statement: Let V, W, X, Y be vector spaces over F and $S, S' : V \to W, T, T' : W \to X, U : X \to Y$ linear transformations. Then

(a) $T(S + S') = TS + TS'$ and $(T + T')S = TS + T'S$
(b) $(UT)S = U(TS)$
(c) $SI_V = I_W S = S$
(d) $a(TS) = (aT)S = T(aS)$ for all $a \in F$.

These can be proved in a similar way to the statements in theorem 2.10.

3.9 Define $U(x, y) = (x, 0)$ and $T(x, y) = (0, x + y)$. Then $UT(a, b) = U(0, a + b) = (0, 0)$ for any (a, b), but $TU(a, b) = T(a, 0) = (a, 0) \neq 0_V$ for $a \neq 0$. By taking the matrix representations of U and T with respect to the standard basis we get $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ respectively. One can check that these matrices satisfy the desired properties.

3.11 Let V be a vector space and $T : V \to V$ be linear. Prove $T^2 = T_0 \iff R(T) \subseteq N(T)$.

$\Rightarrow$ Take $v \in V$ and consider $T^2(v) = T(T(v))$. Since $T(v) \in R(T)$, by assumption, $T(v) \in N(T)$. Therefore $T^2(v) = T(T(v)) = 0$ for all $v \in V$.

$\Leftarrow$ Take $winR(T)$. By definition of the range $w = T(v)$ for some $v \in V$. Then $T(w) = T(T(v)) = T^2(v) = T_0(v) = 0$, so $w \in N(T)$. Thus $R(T) \subseteq N(T)$.

3.15 Let M be an $m \times n$ matrix, and A be a $n \times p$ matrix. Let A_j denote the j^{th} column of A and similarly for $(MA)_j$. Assume $A_j = c_1A_1 + \cdots + c_nA_n$ (where $c_j = 0$) for some j. Thus we have $A_{(k,j)} = c_1A_{(k,1)} + \cdots + c_nA_{(k,n)}$ for all k. By the formula for matrix multiplcation,we have

$$(MA)_{(i,j)} = \sum_{k=1}^{n}(M_{(i,k)}A_{(k,j)}) = \sum_{k=1}^{n}(M_{(i_k)}(c_1A_{(k,1)} + \cdots + c_nA_{(k,n)})$$

$$= \sum_{l=1}^{n} c_l \sum_{k=1}^{n} M_{(i,k)}A_{(k,l)} = \sum_{l=1}^{n} c_l(MA)_{(i,l)}.$$

Thus, the j^{th} column of MA is also a linear combination of the other columns with the same corresponding coefficients.

4.1 (a) False.
(b) True.
(c) False.
(d) False.
(e) True.
(f) False.
(g) True.
(h) True.
(i) True.

4.2 (a) T is not invertible. The dimension of the codomain is larger than the dimension of the domain, so T cannot be onto.
(c) T is invertible. One can check that T is both one-to-one and onto.
(f) T is invertible. One can check that T is both one-to-one and onto.

4.5 Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Proof: If A is invertible then $(A^{-1})^t$ exists. Consider $A^t(A^{-1})^t = ((A^{-1})A)^t = I_n^t = I_n$. Similarly, $(A^{-1})^tA^t = (A(A^{-1}))^t = I_n^t = I_n$. Therefore we get the desired result.

4.10 (a) By exercise (9), if A and B are $n \times n$ and AB is invertible, then A and B are invertible. In this situation, $AB = I_n$ which is an invertible matrix. Therefore A and B are invertible.

(b) Since B is invertible, the B^{-1} exists. By multiplying both sides of the equation on the right by B^{-1} we get $(AB)B^{-1} = A(BB^{-1}) = A(I_n) = A = I_n(B^{-1}) = B^{-1}$.

(c) Let V, W be n dimensional vector spaces, $T : V \to W$, $U : W \to V$ linear transformations. If $UT = I_V$, then T and U are invertible.

Proof: If $UT = I_V$, then $R(U) = V$ and U is onto. Thus, by the Rank-Nullity Theorem, we see that U is one-to-one. Therefore U is invertible. By the same argument as in (b), we can show that $U^{-1} = T$, so T in invertible as well.

5.1 (a) False.

(b) True.

(c) True.

(d) False.

(e) True.

5.2 (a) $Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

(c) $Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

5.6 (a) $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$

(b) $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$

(c) $[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix},$

(d) $[L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix},$

5.10 Assume A and B are similar $n \times n$ matrices, i.e. there exists an invertible matrix Q such that $A = Q^{-1}BQ$. By 2.3, exercise 13, and this identity, $tr(A) = tr((Q^{-1}B)Q) = tr(Q((Q^{-1}B)) = tr(B)$.

5.11 Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α coordinates into β coordinates and β coordinates into γ coordinates, respectively, then RQ is the change of coordinate matrix that changes α coordinates into γ coordinates.

Proof: $RQ = [I_V]_\beta^\gamma [I_V]_\alpha^\beta = [I_V I_V]_\alpha^\gamma$ by theorem 2.11. But $[I_V I_V]_\alpha^\gamma = [I_V]_\alpha^\gamma$, the change of coordinate matrix from α coordinates to γ coordinates.

(b) Prove that if Q changes α coordinates into β coordinates, then Q^{-1} changes β coordinates into α coordinates.

Proof: By theorem 2.11, $[I_V]_\alpha^\beta [I_V]_\beta^\alpha = [I_V]_\alpha^\alpha = I_n$. Since $Q = [I_V]_\alpha^\beta$, by substituting and solving, we get $Q^{-1} = [I_V]_\beta^\alpha$