Assignment 4

MATH110

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Section 1.6: 23, 26, 28, 29, 31; Section 2.1: 1, 4, 6, 9 d e, 10, 12, 14a, 16, 22, 24, 28, 29; Section 2.2: 1, 2 a c f, 4, 5, 9, 13, 14

6.23 (a) $v \in \text{Span}(\{v_1, \cdots, v_k\}) \iff \dim(W_1) = \dim(W_2)$

(b) If $\dim(W_1) \neq \dim(W_2)$, then $\dim(W_1) + 1 = \dim(W_2)$

Proof: Since we have only added one more vector to a generating set of W_2, the dimension can at most increase by one, but we've assumed they aren't equal, so it must increase by at least one. Therefore $\dim(W_2) = \dim(W_1) + 1$.

6.26 The dimension of the subspace $W = \{f \in P_n(R) : f(a) = 0\}$ is n. We prove this by computing a basis.

A basis for this space is $\{x^i - a^i : 1 \leq i \leq n\}$. One can check that this set is linearly independent, has n elements, and the span is contained inside W. If W was bigger, then W would have dimension $n+1$ and be equal to $P_n(R)$, but W is strictly contained in $P_n(R)$. Therefore this set is a basis for W.

6.28 If $\{v_1, \ldots, v_n\}$ is a basis for V over $\mathbb{C}$, then $\{v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n\}$ is a basis for V over (R).

Proof: Linearly Independent. Say $a_1v_1 + b_1iv_1 + \cdots + a_nv_n + a_niv_n = 0$. Then by regrouping we get $(a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n = 0$, but $v_1, \ldots, v_n$ are linearly independent over $\mathbb{C}$ so $a_i + ib_i = 0$ which implies that $a_i = b_i = 0$.
Generate. Take any $v \in V$. Over $\mathbb{C}$, we can write this as $v = (a_1 + ib_1)v_1 + \cdots + (a_n + ib_n)v_n$ for some $a_i + ib_i \in \mathbb{C}$. By distributing we see $v = a_1v_1 + b_1iv_1 + \cdots + a_nv_n + a_niv_n$, so the set generates.

6.29 (a) Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $W_1 \cap W_2$. Extend it to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ for W_1 and to a basis $\{u_1, \ldots, u_k, w_1, \ldots, w_n\}$ for W_2. Then $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ generate. For this set to form a basis we need to check linear independence. We will argue by contradiction. Assume $a_1u_1 + \cdots a_ku_k + b_1v_1 \cdots + b_mv_m + c_1w_1 + \cdots + c_nw_n = 0$, with the constants not all zero. We must have at least some b_i nonzero, because otherwise we have a linear dependence relation among the u_i, w_j and they form a basis. Similarly a c_i must be nonzero. Then we have the relation $a_1u_1 + \cdots a_ku_k + b_1v_1 \cdots + b_mv_m = -c_1w_1 - \cdots - c_nw_n$. But the right hand side of the equation is an element in $W_2 \setminus W_1$ and the right hand side is an element in W_1. Contradiction.

Therefore, $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}$ is a basis for $W_1 + W_2$. So we have $\dim(W_1 + W_2 = k + m + n = (k+m) + (k+n) - k =dim(W_1)$ + dim(W_2) $-$dim$(W_1 \cap W_2)$.

(b) Since V $= W_1 + W_2$, we only need to check that $W_1 \cap W_2 = \{0\}$. Since, by (a), dim(V) $=$ dim$(W_1 + W_2) =$ dim(W_1) + dim(W_2) $-$ dim $(W_1 \cap W_2)$, the intersecion is the zero vector if and only if dim $(W_1 \cap W_2)$=0, which is if and only if dim(V) = dim$(W_1 + W_2)$ = dim(W_1) + dim(W_2).

6.31 (a) Since $W_1 \cap W_2 \subset W_2$, we have dim$(W_1 \cap W_2) \leq$dim$(W_2) = n$.

(b) By [6.29] dim$(W_1 + W_2)$ = dim(W_1) + dim(W_2) $-$dim$(W_1 \cap W_2) \leq$ dim(W_1) + dim$(W_2)$$= m + n$

Chapter 2

1.1 (a) True.

(b) False.

(c) True.

(d) True.

(e) False.

(f) False.

(g) True.

(h) False.

1.4 The null space has dimension 4 and a basis for the null space of T is:

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The dimension of the range of T is 2 and a basis is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Thus we have 4+2 = 6 = dim(V). T is neither one-to-one nor onto.

1.6 The dimension of the null space is $n^2 - 1$. Since the null space is exactly the subspace of trace zero matrices, by last week's homework we know $\{E_{i,j}, E_{1,1} - E_{k,k} : 1 < k \leq n, 1 \leq i, j \leq n, i \neq j\}$ is a basis.

The dimension of the range is 1 and a basis is 1_F.

Thus we have $n^2 - 1 + 1 = n^2 = dim M_{n \times n}(F)$. T is onto.

1.9 (d) $T(-1,0) = (1,0) \neq (-1,0) = -T(1,0)$

(e) $T(0,0) = (1,0 \neq (0,0) = 0_{R^2}$

1.10 $T(2,3) = T((3 \cdot (1,1) - (1,0)) = 3T(1,1) - T(1,0) = (6,15) - (1,4) = (5,11)$. T is one-to-one, because since the dim R(T) = 2, by the rank-nullity theorem, dim N(T) = 0.

1.12 No, there is no such T. By linearity $T(-2,0,6) = -2 \cdot T(1,0,3)$.

1.14a $\Rightarrow$ Assume T does not take linearly independent subsets to linearly independent subsets. This means we can find $v_1, \ldots, v_n$ linearly independent in V, and $a_1, \ldots, a_n$, not all zero, such that $a_1 T(v_1) + \cdots + a_n T(v_n) = 0$. By linearity, $a_1 T(v_1) + \cdots + a_n T(v_n) = T(a_1 v_1 + \cdots + a_n v_n) = 0$. Since $v_1, \ldots, v_n$ are linearly independent, $a_1 v_1 + \cdots + a_n v_n \neq 0$, so T sends a nonzero vector to the zero vector and T is not one-to-one.

$\Leftarrow$ Assume T sends linearly independent sets to linearly independent sets. Take a basis $v_1, \ldots, v_n$ of V. Given a nonzero $v \in V$ write v as $a_1 v_1 + \cdots + a_n v_n$. Then $T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$ Since $T(v_1), \ldots, T(v_n)$ are linearly independent and the a_i are not all zero, $T(v)$ is nonzero. Therefore, T is one-to-one.

1.16 Since T is linear, to show T is onto, it suffices to show that given the basis $1, x, x^2, x^3, \ldots$, there exists an $f_n \in P(R)$ such that $T(f_n) = x^n$ for all n. If we take $f_n = x^{n+1}/(n+1)$, this works. However, T is not one-to-one because $T(c) = 0$ for all constants c.

1.22 Let $a = T(1,0,0), b = T(0,1,0), c = T(0,0,1)$. Then, by linearity, $T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = xa + yb + zc$. For $T : F^n \to F$ you need n scalars, and for $T : F^n \to F^m$ you need n vectors of F^m.

1.24 (a) $T(a,b) = (0,b)$

(b) $T(a,b) = T((0, b-a) + (a,a)) = (0, b-a)$.

1.28 By linearity $T(0_V) = 0_V$, so $\{0\}$ is T-invariant. By definition, $T(V) \subset V$, so V is T-invariant. $T(N(T)) = \{0\} \subset V$ so N(T) is T-invariant. $T(R(T)) \subset R(T)$ by definition of R(T) so R(T) is T-invariant.

1.29 Since W is T-invariant, $T(W) \subset W$, and since W is a subspace, for all $x, y \in W$, $c \in F$, $T(x)+T(y) \subset W$ and $cT(x) \in W$ so T_W is well-defined. Since T is a linear transformation, $W \subset V$, and all properties of the linear transformation hold in V, they must hold in W.

2.1 (a) True.

(b) True.

(c) False. It is an $n \times m$ matrix.

(d) True.

(e) True.

(f) False.

2.2 (a) $\begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$

2.4 $[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

2.5 (a) $[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(c) $[T]_{\alpha^\gamma} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

(d) $[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$

(e) $[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$

(f) $[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

(g) $[a]_\gamma = (a)$

2.9 $T(az_1 + bz_2) = az_1 \bar{+} bz_2 = a\bar{z}_1 + b\bar{z}_2 = a\bar{z_1} + b\bar{z_2} = at(z_1) + bT(z_2)$. So T is linear. $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2.13 Say $aU + bT = 0$. That means for all $v \in V$ $aU(v) + bT(v) = 0_W = U(av) + T(bv)$. Therefore, $U(av) = -T(bv) = T(-bv)$. Since $U(av) \subset R(U)$ and $T(-bv) \subset R(T)$, $U(av) = T(-bv) \subset R(U) \cap R(T) = \{0\}$. Therefore $U(av) = aU(v) = 0 = -bT(v) = T(-bv)$ for all v. Since T and U are nonzero, $a = b = 0$. Thus, U and T are linearly independent.

2.14 Say $a_1T_1 + \cdots + a_nT_n = 0$. Then $(a_1T_1 + \cdots + a_nT_n)(x) = a_1 = 0$. But also $(a_2T_2 + \cdots + a_nT_n)(x^2) = 2a_1 = 0$, so $a_2 = 0$. Repeating this process until x^n will show that all the a_i are zero so the T_i are linearly independent.