Assignment 3

MATH110

January 27, 2007

Section 1.5: 1,3,5,7,10,11,15; Section 1.6: 1,3,5,12,15

5.1 (a) False. There exists a vector in S which is a linear combination of other vectors in S.

(b) True. $c \cdot 0_V = 0_V$ for any nonzero $c \in F$.

(c) False. The empty set is linearly independent.

(d) False. The set consisting of one nonzero vector is linearly independent.

(e) True.

(f) True.

5.3

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{M_{2x3}(F)}$$

5.5 Assume $a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0_{P_n(F)}$. Since the zero vector has negative degree, $a_i = 0$ for all i. Therefore $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

5.7

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generate the 2×2 diagonal matrices.

5.10 $\{(1,0,0), (0,1,0), (1,1,0)\}$ are a linearly dependent set and none are a multiple of the other.

5.11 Span(S) $= \{a_1 u_+ \cdots + a_n u_n | a_i \in Z_2\}$ and because S is a linearly independent set, each vector in Span(S) can be written uniquely in this way. Therefore to count the number of vectors in Span(S), we need to count the possibilities for $(a_1, \ldots, a_n)$. Since Z_2 has 2 elements, there are 2^n possibilities for $(a_1, \ldots, a_n)$, and therefore 2^n vectors in Span(S).

5.15 $S = \{u_1, u_2, \ldots, u_n\}$ Prove S is linearly dependent $\iff$ $u_1 = 0$ or $u_{k+1} \in$Span$(\{u_1, ..., u_k\})$

$\Leftarrow$ This direction follows from the definitions

$\Rightarrow$ Since S is linearly dependent, there exists $a_1, \ldots, a_n \in F$ not all zero, such that $a_1 u_1 + \cdots + a_n u_n = 0$. Let k be such that $a_i = 0$ for all $i > k+1$ and $a_{k+1} \neq 0$. Then our equation reduces to $a_1 u_1 + \cdots + a_{k+1} u_{k+1} = 0$. If $k = 0$, then $u_1 = 0$. If $k > 0$, then $a_1 u_1 + \cdots + a_k u_k = a_{k+1} u_{k+1}$. Since $a_{k+1} \neq 0$, we can divide by a_{k+l} and $u_{k+1} \in$Span$(\{u_1, ..., u_k\})$.

6.1 (a) False. The empty set is a basis for the zero vector space.

(b) True.

(c) False. Infinite dimensional vector spaces do not have finite bases.

(d) False. $\{(1,0), (0,1)\}$ and $\{(1,0), (1,1)\}$ are both bases for $\mathbb{R}^2$.

(e) True.

(f) False. The dimension is $n+1$.

(g) False. The dimension is mn.

(h) True.

(i) False.

(j) True.

(k) True.

(l) True.

6.3 (b), (c), and (d) are bases. (a) and (e) are not.

6.5 No. A set of 4 vectors in a 3-dimensional space is never linearly independent.

6.8 u_1, u_3, u_5, and u_7 form a basis for W.

6.12 Since V is 3-dimensional, it suffices to show that $\{u+v+w, v+w, w\}$ is linearly independent.

$$a(u+v+w)+b(v+w)+cw = 0 = au+(a+b)v+(a+b+c)w.$$

Since $\{u, v, w\}$ are linearly independent, $a = a+b = a+b+c = 0$. Therefore $a = b = c = 0$, and $\{u+v+w, v+w, w\}$ are linearly independent.

6.15 The set $\{E_{i,j}, E_{1,1} - E_{k,k} : 1 < k \leq n, 1 \leq i, j \leq n, i \neq j\}$ is a basis for the trace zero matrices. ($E_{i,j}$ is the matrix with 1 in the i, j place and 0's everywhere else.) The dimension is $n^2 - 1$ because there are $n^2 - 1$ basis vectors.