Solutions

7.4.2.a Since the matrix is given in Jordan canonical form, it's immediate that e_3 generates the generalized 3-eigenspace. Therefore the rational canonical basis is given by e_3, Ae_3, A^2e_3. So, Q is the matrix with these as its columns. And the rational canonical form is $\begin{bmatrix} 0 & 0 & 27 \\ 1 & 0 & -27 \\ 0 & 1 & 9 \end{bmatrix}$.

7.4.2.b A is already in rational canonical form over $\mathbb{R}$.

7.4.2.c Over $\mathbb{C}$ A is diagonalizable, and the rational canonical form is the matrix with eigenvalues on the diagonal, which is $\begin{bmatrix} \frac{1}{2}(-1+i\sqrt(3)) & 0 \\ 0 & \frac{1}{2}(-1-i\sqrt(3)) \end{bmatrix}$.

7.4.3.a Let's write out how T acts on the standard basis.

$$\begin{aligned} T(1) &= x, \\ T(x) &= -1, \\ T(x^2) &= -2, \\ T(x^3) &= -3 \end{aligned}.$$

From this, we can see that the dimension of the image of T is 2, hence the dimension of the nullspace is also 2, i.e. we have a 2 dimensional 0-eigenspace. You can also notice that restricted to the subspace spanned by $\{1, x\}$, T is in rational canonical form (it's rotation by 90 degrees). Thus the rational canonical form is $\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. And the basis that put's T in such a form is $\{1, x, n_1, n_2\}$ where n_i are the basis vectors for the nullspace. The irreducible monic factors are t^2+1 and t.

7.4.3.b The matrix representation of this operator is $A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The characteristic polynomial is $(t^2+1)^2 = t^4+2t^2+1$, and as $T^2 \neq -I$, this is also the minimal polynomial. Therefore the rational canonical form is $\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. β_v, where $v = x\cos x$, puts the operator into rational canonical form.

7.4.5 If the rational canonical form C is diagonal then obviously T is diagonalizable. Now, suppose T is diagonalizable then V is the direct sum of 1-dimensional T-cyclic subspaces. By Thm. 7.17 these form a rational canonical basis. But this basis is a basis of eigenvectors.

7.4.8 Since $\phi(T)$ is not one-to-one, $\exists x$ s.t. $\phi(T)x = 0$. By Ex 7.3.15 the T-annihilator of x divides $\phi(T)$, but since $\phi(T)$ is irreducible it must be equal to the T-annihilator. By part (c) of the same exercise $\phi(T)$ divides the minimal polynomial of T and therefore the characteristic polynomial.

7.4.10 ($\Rightarrow$) Suppose that $x \in C_y$. It follows, since C_y is T-invariant, that $C_x \subseteq C_y$. Now we want to show that $y \in C_x$. Suppose that it's not. Then we can apply the lemma on pg 531, to get that that $\{x, Tx, \ldots, T^k x\} \cup \beta_y$ is a linearly independent set. But, the hypothesis states that $x \in span\beta_y$, a contradiction. Thus $y \in C_x$. ($\Leftarrow$) is immediate.

* Assume the minimal polynomial of T, denoted by $\mu_T(t) = \prod \phi_k$ has distinct irreducible factors. Therefore $N(\phi_k T) = K_{\phi_k}$. First, for a fixed i, suppose that $deg(\phi_i) = 1$. Thus, K_{ϕ_i} is some generalized eigenspace K_λ. Now pick $v \in K_{\phi_i}$. Since $\mu_T(T) = 0$, $0 = \mu_T(T)v = \prod \phi_k(T)v$. So $\phi_j(T)v = 0$ for some j. However since $\phi_k(T)|_{K_{\phi_i}}$ is 1-1 when $k \neq i$, it follows that $j = i$ and so $0 = \phi_i(T)v = (T - \lambda I)v$, i.e. $v \in E_\lambda$ and so $K_{\phi_i} = E_\lambda$. Therefore, as E_λ is the direct sum of 1-dimensional invariant subspaces (i.e. the ones spanned by the eigenvectors), it follows that K_{ϕ_i} is semisimple. Now, if ϕ_i has degree greater than one, we claim that K_{ϕ_i} is simple. If W is a non-zero invariant subspace, then $\mu_{T|_W}$ divides ϕ_i, but as ϕ_i is irreducible, we have that $\mu_{T|_W} = \phi_i$ ($\mu_{T|_W}$ isn't constant, as W is non-zero.) Now if $x \in K_{\phi_i}$, and $x \notin W$, we again use the lemma on pg. 351 to get another invariant subspace $W \bigoplus C_x \subseteq K_{\phi_i}$. However, the minimal polynomial of T restricted to this subspace divides $\mu_{T|_W} = \phi_i$, and therefore $\mu_{T|_W} \cdot \mu_{T|_{C_x}}$ divides $\mu_{T|_W}$. This forces $\mu_{T|_{C_X}}$ to be a constant, a contradiction. Thus no $x \in K_{\phi_i}$ can lie outside of W – i.e. $K_{\phi_i} = W$, and is therefore simple. Now it's easy to verify that the direct sum of all these simple subspaces is V.