Math 110, Professor Ogus

7.3.1 (a)F (b)T (c) F (d) F (e) T (f) F (g) T (h) T (i) T

7.3.2 (a) The characteristic polynomial is $(t-1)(t-3)$, which has distinct roots, since the roots of the minimal polynomial are exactly the eigenvalues, the two must coincide.

7.3.2 (b) The characteristic polynomial is $(t-1)^2(t-2)$, so we check if $(t-1)(t-2)$ is the minimal polynomial. Expanding this, it amounts to checking if $A^2 - 3A + 2I = 0$. But this is not zero, and so the minimal polynomial must be $(t-1)^2(t-2)$.

7.3.3(a),7.3.4(a) With respect to the standard basis, $[T] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the characteristic polynomial is $(t+\sqrt{2})(t-\sqrt{2})$, which has distinct roots – thus it is also the minimal polynomial. This also implies the operator is diagonalizable.

7.3.3(d),7.3.4(d) First notice that $T^2 = I$, so $p(T) = 0$, where $p(t) = t^2 - 1$. The minimal polynomial divides $p(t) = (t-1)(t+1)$, but since $T \pm I \neq 0$, $p(t)$ must be the minimal polynomial. To show diagonalizability, we present a basis of eigenvectors: $E_1 = span\{E_{11}, E_{22}, E_{12} + E_{21}\}$ and $E_{-1} = span\{E_{12} - E_{21}\}$.

7.3.8 (a) $T$ is invertible iff 0 is not an eigenvalue. But the eigenvalues are precisely the roots of it's minimal polynomial. Thus $T$ is invertible iff $p(0) \neq =$.
(b) we have $0 = p(T) = T^n + a_{n-1}T^{n-1} + \ldots + a_0 I$, and by moving the last term to the other side, and dividing my $a_0$, we get $I = \frac{-1}{a_0}(T^n + a_{n-1}T^{n-1} + \ldots + a_1 T)$, pulling out $T$, we get $I = \frac{-1}{a_0}(T^{n-1} + a_{n-1}T^{n-2} + \ldots + a_1 I)T$ – multiplying on the right by $T^{-1}$ gives the result.

7.3.9 Since $T$ is diagonalizable, we have that $V$ is the direct sum of the eigenspaces $E_{\lambda_i}$, and thus every vector in $V$ has a unique representation as the sum of eigenvectors. Now $V$ is $T$-cyclic iff $\exists x \in V$ s.t. $\{x, Tx, \ldots, T^{n-1}x\}$ is a basis for $V$. Now, using the direct sum decomposition, write $x = v_1 + \ldots + v_k$. So we therefore have the basis $\{v_1 + \ldots + v_k, \lambda_1 v_1 + \cdots + \lambda_k v_k, \ldots, \lambda_1^{n-1} v_1 + \ldots + \lambda_k^{n-1} v_k\}$ Thus a general vector $w = a_0(v_1 + \ldots + v_k) + a_1(\lambda_1 v_1 + \cdots + \lambda_k v_k) + \ldots + a_{n-1}(\lambda_1^{n-1} v_1 + \ldots + \lambda_k^{n-1} v_k)$. Or, if we call $g(t) = a_0 + a_1 t + \ldots + a_{n-1} t^{n-1}$, then we can write $w = g(\lambda_1)v_1 + \ldots + g(\lambda_k)v_k$ – i.e. $\{v_1, \ldots, v_k\}$ is a basis for $V$ – i.e. each $E_{\lambda_i}$ is one-dimensional. For the other direction, we note that this argument is reversible.

7.3.12 Suppose $\exists g(t)$ s.t. $g(D) = 0$. if $deg(g) = n$, then we look at $g(D)x^n$. $0 = g(D)x^n = a_n D^n x^n + \ldots + a_0 x^n = a_n n! + \ldots + a_0 x^n$, but for this to be zero as a polynomial, all the coefficients must be zero – i.e. $g(t) = 0$. This, however, cannot be a minimal polynomial, as it's not monic.

7.3.15 (a)The $T$-annihilator always exists, as the minimal polynomial annihilates $x$. Now suppose $p, q$ are both $T$-annihilators – so both have the same degree (as they're both least degree) and both are monic – but $(p - q)(T)x = 0$ and has a strictly smaller degree, and we can make it monic by dividing through by the next nonzero coefficient. This contradicts the fact that $p$ and $q$ have least degree, and so thus there must not be another nonzero coefficient – i.e. $p = q$. (b) The proof for Theorem 7.12(a) works verbatim for this. (c) We need to show that $p(T_W) = 0$. As $W$ is $T$-cyclic, any $w = q(T)x$ for some polynomial $q$. Now, $p(T_W)w = p(T)w = p(T)q(T)x = q(T)P(T)x = q(T)0 = 0$. $p(t)$ is also of least degree, as if there was a polynomial of smaller degree that killed all of $W$, then it would also send $x$ to zero, as $x \in W$. (d) $p(t) = t - \lambda$, then $p(T)x = (T - \lambda)x = 0$ iff $x \in E_\lambda$.