Math 110, Professor Ogus, Homework due 4/11

6.4.1 (a)T (b)F (c) F (d) T (e) T (f) T (g) F (h) T

6.4.3 1(c) is true if the basis is an orthonormal basis. Thus we need to find a normal operator whose matrix representation with respect to a NON-orthogonal basis is not normal. Let $T = L_A$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Since $T^* = -T$, T is normal. However, if we consider the basis $\beta = \{(1,0),(1,1)\}$, $[T]_\beta = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$, which is not normal.

6.4.4 Assuming T and U are self-adjoint, we have TU are self adjoint $\Leftrightarrow (TU)^* = TU \Leftrightarrow U^*T^* = TU \Leftrightarrow UT + TU$.

6.4.6 (a) This exercise shows that every operator has a decomposition into a sum of a self-adjoint operator (T_1) and a skew-adjoint operator (iT_2) – namely $T = T_1 + iT^2$. $T_1^* = \frac{1}{2}(T+T^*)^* = \frac{1}{2}(T^* + T^{**}) = \frac{1}{2}(T^* + T) = T_1$ $T_2^* = (\frac{1}{2i}(T - T^*))^* = \frac{-1}{2i}(T - T^*)^* = \frac{-1}{2i}(T^* - T^{**}) = T_2$ $T_1 + iT_2 = \frac{1}{2}(T+T^*) + i\frac{1}{2i}(T - T^*) = \frac{2T}{2} = T$ (b) This exercise shows that such a decomposition is unique: given self-adjoint operators U_1 and U_2 such that $T = U_1 + iU_2$, we show that $U_1 = T_1$ and $U_2 = T_2$. First we calculate $T_1 = \frac{1}{2}(T+T^*) = \frac{1}{2}((U_1+iU_2)+(U_1+iU_2)^*) = \frac{1}{2}(U_1+iU_2+U_1^*-iU_2^*) = U_1$ $T_2 = \frac{1}{2i}(T-T^*) = \frac{1}{2i}((U_1+iu_2)-(U_1+iU_2)^*) = \frac{1}{2i}(U_1+iU_2-U_1+iU_2^*) = U_2$ (c) First we calculate $T^*T = (T_1 - iT_2)(T_1 + iT_2) = (T_1^2 + T_2^2) + i(T_1T_2 - T_2T_1)$ $TT^* = (T_1 + iT_2)(T_1 - iT_2) = (T_1^2 + T_2^2) + i(T_2T_1 - T_1T_2)$ Thus, setting the above equations equal and simplifying, we see that $T^*T = TT^* \Leftrightarrow T_1T_2 = T_2T_1$.

6.4.7 (a) Given $T = T^*$ we show that $T|_W = (T|_W)^*$ – i.e. that $\langle T|_W x, y\rangle = \langle x, T|_W y\rangle$ $\forall x, y \in W$. But this equality holds, as $x, y \in V$ and T is self-adjoint as an operator on V.
(b)We need to show $T^*(W^\perp) \subseteq W^\perp$. Fix an element $v \in W^\perp$. Then $\forall x \in W$ we have $\langle x, T^*v\rangle = \langle Tx, v\rangle = 0$ where the first equality holds by (a) and the second equality holds because $v \in W^\perp$ and $T(w) \in W$. Thus $T^*(v) \in W^\perp$.
(c)We need the hypotheses that W is $T-$ and T^*-invariant so that the operators $T|_W$ and $(T^*)|_W$ are well-defined operators on W. Now $\forall x, y \in W$ $\langle x, (T|_W)^* y\rangle|_W = \langle T|_W x, y\rangle|_W = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, (T^*)|_W y\rangle|_W$. The above comment about these operators being well-defined operators on W guarantees that all the entries in these inner-products lie in W.
(d) $T|_W T|_W^* = (T|_W)(T^*|_W)$ by part (c). Because T is a normal operator in V, we have $(T|_W)(T^*|_W) = (TT^*)|_W = (T^*T)|_W = (T^*|_W)(T|_W)$. Then, again by part (c), we have $(T^*|_W)(T|_W) = (T|_W)^*(T|_W)$.

6.4.8 First we need to prove Ex 5.4.24: if T is a diagonalizable operator on V (finite dimensional), then $T|_W$ is diagonalizable for any T-invariant subspace W. Pf: If T is diagonalizable, then V is a direct sum of the eigenspaces of T. So given (a basis element of W) $v \in W$ we can write it as a linear comination of distinct eigenvectors $v = a_1v_1 + \ldots + a_kv_k$. By Midterm problem 4 (and a simple induction argument), we have that that each of these $v_k \in W$. Thus the set of all such v_k span W, and so by the Replacement Theorem, we can pick a basis for W among the v_k. These are all eigenvectors of T and therefore also eigenvectors of $T|_W$. Thus this basis is in fact a basis of eigenvectors, and so $T|_W$ is diagonalizable. [done with pf]. Now to prove the problem: assume that W is T-invariant, and that T is a normal operator on a finite dimensional complex inner product space. Since T is normal, T is diagonalizable. From the lemma above, $T|_W$ is also diagonalizable, and we therefore get a basis of W consisting of eigenvectors. By theorem 6.15, an eigenvector of $T|_W$ is also an eigenvector of $T^*|_W$, and so each eigenspace is also an eigenspace for $T^*|_W$. Since eigenspaces are invariant subspaces, we have that W is T^*-invariant.

6.4.9 Theorem 6.15 gives $\|Tx\| = \|T^*x\|$. Thus, $x \in N(T) \Leftrightarrow \|T^*x\| = \|Tx\| = \|0\| = 0$. As inner products are positive definite, this happens precisely when $T^*x = 0$ – i.e. when $x \in N(T^*)$. Thus we have $N(T) = N(T^*)$.
By Ex 6.3.12, we have that $R(T^*)^\perp = N(T)$. Since, T is an operator on a finite dimensional inner product space, we get that $R(T^*) = R(T^*)^{\perp\perp} = N(T)^\perp$. But, by the first part of this problem, $N(T)^\perp = N(T^*)^\perp = R(T^{**}) = R(T)$. Thus $R(T^*) = R(T)$.

6.4.12 One way to approach this problem is to apply Shur's theorem to get a basis β in which $[T]_\beta$ is upper-triangular, and proof of theorem 6.16 works verbatim.

6.4.22(b) We assume that $\langle x, y\rangle' = \langle Tx, y\rangle$ is an inner product. First we show that T is positive definite w/r/t the standard inner product: $\langle Tx, x\rangle = \langle x, x\rangle' \geq 0$ because we're assuming $\langle \cdot, \cdot\rangle'$ is an inner-product (and is therefore positive definite). We now show that T is positive definite w/r/t the inner product defined in the problem: $\langle Tx, x\rangle' = \langle T^2x, x\rangle = \langle Tx, Tx\rangle \geq 0$. The second equality used that T is self-adjoint (w/r/t the standard inner-product), which we must now prove. $\langle Tx, y\rangle = \langle x, y\rangle' = \overline{\langle y, x\rangle'} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$ – Thus $T = T^*$.

6.5.1 (a) T (b) F (c) F (d) T (e) F (f) T (g) F (h) F (i) F

6.5.2(a) A is self-adjoint and is therefore unitarily diagonalizable (and so the change of basis matrix that takes A to D will be unitary). A has eigenvalues $\lambda = -1, 3$. The (unit length) eigenvector corresponding to -1 is $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. The eigenvector corresponding to 3 is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. While $D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

6.5.2(e) A is self-adjoint (and thus normal) and so is unitarily diagonalizable. A has eigenvalues $\lambda = 8, -1$. The unit length eigenvector corresponding to 8 is $(\frac{1}{\sqrt{3}}, \frac{1+i}{\sqrt{3}})$, while the eigenvector corresponding to -1 is $(-\frac{2}{\sqrt{6}}, \frac{1+i}{\sqrt{6}})$. Thus $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \end{pmatrix}$, while $D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$

6.5.5 (a) These are NOT unitarily equivalent, as they have different eigenvalues. (b) These are NOT unitarily equivalent, as they have different eigenvalues. (c) These are NOT unitarily equivalent, as they have different eigenvalues. (d) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is normal and thus unitarily equivalent to a diagonal matrix. Furthmore, as the two matrice's eigenvalues coincide, the diagonal matrix is the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$. Thus the two matrices are unitarily equivalent. (e) These are NOT unitarily equivalent. The first matrix is not self-adjoint, and is therefore not unitarily equivalent to a real diagonal matrix – while the second matrix is a real diagonal matrix.

6.5.10 Since A is real-symmetric or complex-normal, we know that A is diagonalizable. i.e. $A = PDP^{-1}$ where $D_{ij} = 0$ if $i \neq j$ and $D_{ii} = \lambda_i$, where λ_i are the eigenvalues of A. Thus we have $tr(A) = tr(PDP^{-1}) = tr(DP^{-1}P) = tr(D) = \sum \lambda_i$. Next (remembering that $P^* = P^{-1}$) we have $A^* = (PDP^*)^* = PD^*P^{-1}$. $tr(A^*A) = tr(PD^*P^{-1}PDP^{-1}) = tr(D^*D) = \sum |\lambda_i|^2$.

6.5.15 (a)We have that $U(W) \subseteq W$ because W is an invariant subspace. However, U is unitary, and therefore invertible, so $N(U|_W) = \{0\}$ and so $Rank(U|_W) = dimW$ – which implies that $U(W) = W$.
(b) Suppose that $v \in W^\perp$ (so $\langle v, w \rangle = 0\ \forall w \in W$). We have $\langle Uv, w \rangle = \langle v, U^* w \rangle = \langle v, U^{-1} \rangle = 0$ because U restricts to an invertible operator on W (with $(U^{-1})|_W = (U|_W)^{-1}$, by 6.4.7(c)) and so $U^{-1}w \in W$. Thus $Uv \in W^\perp$.

6.5.16 Let V the inner product space of double infinite sequences with only a finite number of nonzero terms. i.e. $V = \{(\ldots, \sigma(k), \sigma(k+1), \ldots); \sigma(k) = 0$ for all but a finite number of $k\}$ with inner product given by $\langle \sigma, \mu \rangle = \sum_{k \in Z} \sigma(k)\overline{\mu(k)}$. Let $T(\sigma)(k) = \sigma(k+1)$ be the left shift operator. T is an isometry and is surjective, so is therefore unitary. Fix a number N, and consider the invariant subspace $W = \{\sigma;\ \sigma(k) = 0\ \forall n \geq N\}$. We calculate $W^\perp$ using the following basis for W: $\{e_i\}_{i \leq N}$ where $e_i(k) = \delta_{ik}$. So if $\sigma \in W^\perp$, then $0 = \langle \sigma, e_i \rangle = \sigma(i)$, $\forall i \leq N$. Since this condition is also sufficient to be in $W^\perp$, we see $W^\perp = \{\sigma;\ \sigma(k) = 0\ \forall k \leq N\}$. This is NOT invariant under left shift. NOTE: compare this example to example 3 on pg 372.