Math 110, Professor Ogus, Homework due 4/11

6.2.1 (a)F (b)T (c) T (d) F (e) F (f) F (g) F (h) T

6.2.6 Let $W \subseteq V$ and $x \notin W$. We can write any $w \in W$ in the following way: $w = w + (x - w)$. According to theorem 6.6, there is a unique such $w \in W$ such that $(x - w) \in W^{\perp}$. This gives $0 = \langle x - w, w\rangle = \langle x, w\rangle - \langle w, w\rangle$, i.e. $\langle x, w\rangle = \|w\|^2$. Now

$$\begin{aligned}\langle x, x - w\rangle &= \langle x, x\rangle - \langle x, w\rangle &(1)\\ &= \langle x, x\rangle - \langle w, w\rangle &(2)\\ &= \|x\|^2 - \|w\|^2 &(3)\\ &= \|x - w\|^2 > 0 &(4)\end{aligned}$$

Where the last equality holds by the Pythagorean Theorem, and the inequality holds because inner products are positive definite (and $x \notin W$). Thus, taking $y = x - w$ yields the result.

6.2.7 ($\Rightarrow$) if $z \in W^{\perp}$ then $\langle z, v\rangle = 0\ \forall v \in \beta$ as each $v \in W$.
($\Leftarrow$) Let $w \in W$ and write $w = \sum a_i v_i$. Now, $\langle z, w\rangle = \langle z, \sum a_i v_i\rangle = \sum \bar{a_i}\langle z, v_i\rangle = \sum \bar{a_i} \cdot 0 = 0$, where we deduce $\langle z, v_i\rangle = 0$ by hypothesis. Thus $z \in W^{\perp}$, as desired.

6.2.9 If $W = span\{(i, 0, 1)\} \subseteq C^3$, then an orthonormal basis for W is $\{\frac{1}{\sqrt{2}}(i, 0, 2)\}$. For finding an orthonormal basis of $W^{\perp}$, we need to find 2 orthogonal unit vectors that satisfy $\langle x, y, z\rangle \cdot \langle i, 0, 1\rangle = 0$. One can notice that $\{\frac{1}{\sqrt{2}}(1, 0, -i), (0, 1, 0)\}$ works.

6.2.10 Fix a basis $\beta = \{v_1, \ldots, v_k\}$, for W and define $T(y) = \sum\langle y, v_i\rangle v_i$. T is linear, as it sends basis elements to basis elements, and is a projection by theorem 6.6. Again, by theorem 6.6, each vector $y \in V$ can be uniquely written as $y = T(y) + z$, where $T(y) \in W$ and $z \in W^{\perp}$. Thus we see that $T(y) = 0$ iff $y \in W^{\perp}$ i.e. $N(T) = W^{\perp}$.
Lastly, we again use the decomposition $y = T(y) + z$ to obtain $\|T(y)\|^2 \leq \|T(y)\|^2 + \|z\|^2 = \|T(y) + z\|^2 = \|y\|^2$, where the middle equality follows from the Pythagorean theorem. Taking square roots gives the result.

6.2.19(a) Given $W = \{(x, y) \in R^2; y = 4x\}$, an orthonormal basis for W is $\{\frac{1}{\sqrt{17}}(1, 4)\}$. From the previous problem, the orthogonal projection is $T(u) = \sum\langle u, v_i\rangle v_i$. Thus $T(u) = \langle u, v\rangle v = \langle (2, 6), \frac{1}{\sqrt{17}}(1, 4)\rangle \frac{1}{\sqrt{17}}(1, 4) = \frac{1}{\sqrt{17}}(2{+}24)(\frac{1}{\sqrt{17}}(1, 4)) = (\frac{26}{17}, \frac{104}{17})$.

6.2.19(b) An orthonormal basis for $W = P_1(R)$ is $\{1, 2\sqrt{3}(x - \frac{1}{2})\}$ (see previous HW). Thus the projection is $T(h) = \langle h(x), 2\sqrt{3}(x - \frac{1}{2})\rangle(2\sqrt{3})(x - \frac{1}{2}) + \langle h(x), 1\rangle 1$. We calculate $\langle h(x), 2\sqrt{3}(x - \frac{1}{2})\rangle = 2\sqrt{3}\int_0^1 (x - \frac{1}{2})(4 + 3x -$

$2x^2)dx = \frac{\sqrt{3}}{6}$, and $\langle h(x), 1\rangle = \int_0^1 4 + 3x - 2x^2 = \frac{29}{3}$. The projection is therefore equal to $(\frac{\sqrt{3}}{6})(2\sqrt{3}(x - \frac{1}{2})) + \frac{29}{6} = x - \frac{13}{3}$

6.2.20 (a) By the corollary to theorem 6.6, the distance is $\|T(u) - u\| = \|(\frac{26}{17}, \frac{104}{17}) - (2, 6)\| = \frac{2}{\sqrt{17}}$
(b) As above, the distance is $\|T(h) - h\|$. . .

6.2.30 (a) That $\langle \sigma_1 + c\sigma_2, \mu\rangle = \langle \sigma_1, \mu\rangle + c\langle \sigma_2, \mu\rangle$, follows from the field axioms (in particular, associativity of multiplication and the distributive law). Next, we see $\overline{\langle \sigma, \mu\rangle} = \overline{\sum \sigma(n)\overline{\mu(n)}} = \sum \overline{\sigma(n)}\overline{\overline{\mu(n)}} = \sum \mu(n)\overline{\sigma(n)} = \langle \mu, \sigma\rangle$. Lastly, we check that it's positive definite: $\langle \sigma, \sigma\rangle = \sum \sigma(n), \overline{\sigma(n)} = \sum |\sigma(n)|^2 > 0$ exactly when $\sigma \neq 0$.
(b) $\langle e_i, e_j\rangle = \sum \delta_{ik}\delta_{jk} = \delta_{ij}$ and so the $\{e_i\}$ form an orthonomal set. Furthermore, they form a basis, as any sequence σ can be uniquely written as $\sum \sigma(n)e_n$.
(c) Define $\sigma_n = e_1 + e_n$ and $W = span(\{\sigma_n; n \geq 2\})$. First we show that $e_1 \notin W$. Suppose, for the sake of contradiction, that $e_1 \in W$. Then we can find scalars a_i with $i \geq 2$ such that $e_1 = \sum a_i\sigma_i = \sum a_i(e_i + e_i) = \sum a_ie_1 + \sum a_ie_i$. Rearranging the terms we get $(\sum(a_i - 1))e_1 + \sum a_ie_i = 0$ – as the $\{e_i\}$ are linearly independent, the a_i must all be zero, contradicting that $\sum a_i - 1 = 0$. Thus $W \neq V$. Now we show that $W^\perp = \{0\}$: if $\langle \sigma_n, \mu\rangle = 0\ \forall n \in N$, then $\sum(e_1 + e_n)(k)\overline{\mu(k)} = 0$. i.e. $\sum_k e_1(k)\overline{\mu(k)} = -\sum_k e_n(k)\overline{\mu(k)}$. This holds for every n exactly when $\mu(1) = -\mu(n)$ $\forall n \in N$. Such a μ is an element of W iff $\mu(n) = 0\ \forall n \in N$ (otherwise, the sequence would have infinitely many nonzero terms). Thus $W^\perp = \{0\}$ and so $(W^\perp)^\perp = \{0\}^\perp = V \neq W$.

6.3.1 (a)T (b)F (c) F (d) T (e) F (f) T (g) T

6.3.2(b) We want find (y_1, y_2) such that $z_1 - 2z_2 = g(z_1, z_2) = \langle (z_1, z_2), (y_1, y_2)\rangle = z_1\overline{y_1} + z_2\overline{y_2}$. We see that $(1, -2)$ works.

6.3.3(a) We see that with respect to the standard basis, $[T] = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$, which is equal to its conjugate transpose. We have that $[T^*] = [T]^* = [T]$ and so the operator is self-adjoint.

6.3.3(b) With respect to the standard basis, $[T] = \begin{pmatrix} 2 & i \\ 1-i & 0 \end{pmatrix}$ and so $[T^*] = [T]^* = \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}$. Thus, we see $T(z_1, z_2) = (2z_1 + (1+i)z_2, -iz_1)$

6.3.6 Given $U_1 = T + T^*$, we calculate $U_1^* = (T + T^*)^* = T^* + T^{**} = T^* + T = T + T^* = U_1$
Given $U_2 = TT^*$ we calculate $U_2^* = (TT^*)^* = (T^*)^*T^* = TT^* = U_2$.

6.3.9 If $V = WW^\perp$ and T is the projection on W along $W^\perp$, we see that $T|_W = Id$ and $T|_{W^\perp} = 0$. So if we choose a basis for V, $\beta = \{v_1, \ldots, v_k, w_1, \ldots, w_m\}$

where $\{v_i\}$ is a basis for W and $\{w_i\}$ is a basis for $W^\perp$. Then we see $[T] = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, which is self-adjoint. Therefore, by Thm 6.10, T is self-adjoint.

6.3.10 ($\Leftarrow$) This follows from taking $x = y$.
($\Rightarrow$) We use the polar identity:

$$\begin{aligned}
\langle x, y \rangle &= \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2 & (5)\\
&= \frac{1}{4}\sum i^k \|T(x) + i^k T(y)\|^2 & (6)\\
&= \frac{1}{4}\sum i^k \langle T(x) + i^k T(y), T(x) + i^k T(y)\rangle & (7)\\
&= \frac{1}{4}\sum i^k (\|T(x)\|^2 + \|T(y)\|^2 + 2\Re\langle T(x), i^k T(y)\rangle) & (8)\\
&= \frac{1}{4}\sum 2\Re\langle T(x), i^k T(y)\rangle & (9)\\
&= \langle T(x), T(y)\rangle & (10)
\end{aligned}$$

Where we go from (5) to (6) using the hypothesis, and (6) to (7) by the definition of the norm. If you're having trouble going from (9) to (10), write $\langle T(x), T(y)\rangle = a + ib$ (so $\Re\langle T(x), T(y)\rangle = a$ and $\Im\langle T(x), T(y)\rangle = b$) and write out each of the terms of the sum in terms of a and b. . .

6.3.12(a) Pick $x \in R(T^*)^\perp$, then $\forall T^*y \in R(T^*)$ we have $0 = \langle x, T^*y\rangle = \langle Tx, y\rangle$. Since this holds for all y, we can take $y = Tx$ to get $\langle Tx, Tx\rangle = 0$. Thus $Tx = 0$ and so $x \in N(T)$. Now, pick $x \in N(T)$. Then $0 = \langle Tx, y\rangle = \langle x, T^*y\rangle\ \forall y$, i.e. $x \in R(T^*)^\perp$.

6.3.12(b) This follows immediately from the proposition that states: if W is a finite dimensional subspace of an inner product space, then $(W^\perp)^\perp = W$. Proof: pick $x \in W$, then it's immediate that $\langle x, y\rangle = 0\ \forall y \in W^\perp$, and thus $x \in (W^\perp)^\perp$. Now, Suppose $x \in (W^\perp)^\perp$. Now if $x \notin W$, 6.2.6 gives the existence of a $y \in W^\perp$ such that $\langle x, y\rangle \neq 0$ – i.e. $x \notin (W^\perp)^\perp$, a contradiction.

6.3.13(a) That $N(T) \subseteq N(T^*T)$, follows from the fact that $T(0) = 0$. We now prove $N(T^*T) \subseteq N(T)$: pick $x \in N(T^*T)$, so $Tx \in N(T^*) = R(T)^\perp$. Thus $\forall v \in V$ we have $\langle Tx, Tv\rangle = 0$. Taking $v = x$ we get $\langle Tx, Tx\rangle = 0$ – i.e. $Tx = 0$.
Now since the nullspaces are equal, it follows that their nullities are equal. Thus, by the dimension theorem, their Ranks are equal.

6.3.13(b) Note that $R(T^*) = N(T)^\perp$ and so $Rk(T^*) = dim(N(T)^\perp) = n - dim N(T) = Rk(T)$. The second equality follows from Theorem 6.7(c), and the third equality is the dimension theorem. Thus we have that $Rk(TT^*) = Rk((T^*)^*T^*) =$

$Rk(T^*) = Rk(T) = Rk(T^*T)$. The second and fourth equalities are from part (a).

6.3.13(c) This follows immediately from $Rk(L_A) = Rk(A)$ and $[L_A^*] = A^*$.