Math 110, Professor Ogus, Homework due 4/11

6.2.1 (a) F (b) T (c) T (d) F (e) F (f) F (g) F (h) T

6.2.6 Let $W \subseteq V$ and $x \notin W$. We can write any $w \in W$ in the following way:

$$w = w + (x - w).$$

According to theorem 6.6, there is a unique such $w \in W$ such that $(x - w) \in W^\perp$. This gives $0 = \langle x - w, w \rangle = \langle x, w \rangle - \langle w, w \rangle$, i.e. $\langle x, w \rangle = \|w\|^2$. Now

$$\langle x, x - w \rangle = \langle x, x \rangle - \langle x, w \rangle \quad (1)$$

$$= \langle x, x \rangle - \langle w, w \rangle \quad (2)$$

$$= \|x\|^2 - \|w\|^2 \quad (3)$$

$$= \|x - w\|^2 > 0 \quad (4)$$

Where the last equality holds by the Pythagorean Theorem, and the inequality holds because inner products are positive definite (and $x \notin W$). Thus, taking $y = x - w$ yields the result.

6.2.7 ($\Rightarrow$) if $z \in W^\perp$ then $\langle z, v \rangle = 0 \forall v \in \beta$ as each $v \in W$.

($\Leftarrow$) Let $w \in W$ and write $w = \sum a_i v_i$. Now, $\langle z, w \rangle = \langle z, \sum a_i v_i \rangle = \sum a_i \langle z, v_i \rangle = \sum a_i - 0 = 0$, where we deduce $\langle z, v_i \rangle = 0$ by hypothesis. Thus $z \in W^\perp$, as desired.

6.2.9 If $W = \text{span}\{ (i, 0, 1) \} \subseteq C^3$, then an orthonormal basis for $W$ is \left\{ \frac{1}{\sqrt{2}}(i, 0, -i), (0, 1, 0) \right\}.

For finding an orthonormal basis of $W^\perp$, we need to find 2 orthogonal unit vectors that satisfy $\langle x, y, z \rangle \cdot (i, 0, 1) = 0$. One can notice that \left\{ \frac{1}{\sqrt{2}}(1, 0, -i), (0, 1, 0) \right\} works.

6.2.10 Fix a basis $\beta = \{ v_1, \ldots, v_k \}$, for $W$ and define $T(y) = \sum \langle y, v_i \rangle v_i$. $T$ is linear, as it sends basis elements to basis elements, and is a projection by theorem 6.6. Again, by theorem 6.6, each vector $y \in V$ can be uniquely written as $y = T(y) + z$, where $T(y) \in W$ and $z \in W^\perp$. Thus we see that $T(y) = 0$ iff $y \in W^\perp$ i.e. $N(T) = W^\perp$.

Lastly, we again use the decomposition $y = T(y) + z$ to obtain $\|T(y)\|^2 \leq \|T(y)\|^2 + \|z\|^2 = \|T(y) + z\|^2 = \|y\|^2$, where the middle equality follows from the Pythagorean theorem. Taking square roots gives the result.

6.2.19(a) Given $W = \{ (x, y) \in R^2; y = 4x \}$, an orthonormal basis for $W$ is \left\{ \frac{1}{\sqrt{17}}(1, 4) \right\}.

From the previous problem, the orthogonal projection is $T(u) = \sum \langle u, v_i \rangle v_i$.$$

Thus $T(u) = \langle u, v \rangle v = \langle (2, 6), \frac{1}{\sqrt{17}}(1, 4) \rangle \frac{1}{\sqrt{17}}(1, 4) = \frac{1}{\sqrt{17}}(2 + 24)\frac{1}{\sqrt{17}}(1, 4) = \frac{26}{17}$, $\frac{104}{17}$.

6.2.19(b) An orthonormal basis for $W = P_1(R)$ is \{1, $2\sqrt{3}(x - \frac{1}{2})$\} (see previous HW). Thus the projection is $T(h) = \langle h(x), 2\sqrt{3}(x - \frac{1}{2}) \rangle (2\sqrt{3})(x - \frac{1}{2}) + \langle h(x), 1 \rangle 1$. We calculate $\langle h(x), 2\sqrt{3}(x - \frac{1}{2}) \rangle = 2\sqrt{3} \int_0^1 (x - \frac{1}{2})(4 + 3x -$
\[2x^2 dx = \frac{\sqrt{3}}{6},\text{ and } \langle h(x), 1 \rangle = \int_0^1 4 + 3x - 2x^2 = \frac{29}{3}.\text{ The projection is therefore equal to } (\frac{\sqrt{3}}{6})(2\sqrt{3}(x - \frac{1}{2})) + \frac{29}{6} = x - \frac{13}{3}.

6.2.20 \text{ (a) By the corollary to theorem 6.6, the distance is } \|T(u) - u\| = \|(\frac{29}{36}, \frac{104}{17}) - (2, 6)\| = \frac{29}{\sqrt{17}}.
\text{ (b) As above, the distance is } \|T(h) - h\|.

6.2.30 \text{ (a) That } \langle \sigma_1 + a\sigma_2, \mu \rangle = \langle \sigma_1, \mu \rangle + c\langle \sigma_2, \mu \rangle \text{, follows from the field axioms (in particular, associativity of multiplication and the distributive law). Next, we see } \langle \sigma, \mu \rangle = \sum_{n \in N} \sigma(n)\mu(n) = \sum_{n \in N} \sigma(n)\mu(n) = \langle \sigma, \mu \rangle. \text{ Lastly, we check that it’s positive definite: } \langle \sigma, \sigma \rangle = \sum_{n \in N} |\sigma(n)|^2 > 0 \text{ exactly when } \sigma \neq 0.
\text{ (b) } \langle e_i, e_j \rangle = \sum_{n \in N} \delta_{ik}\delta_{jk} = \delta_{ij} \text{ and so the } \{e_i\} \text{ form an orthonomal set. Furthermore, they form a basis, as any sequence } \sigma \text{ can be uniquely written as } \sum_{n \in N} \sigma(n)\epsilon_n.
\text{ (c) Define } \sigma_n = e_1 + e_n \text{ and } W = \text{span}(\{\sigma_n : n \geq 2\}). \text{ First we show that } e_1 \notin W. \text{ Suppose, for the sake of contradiction, that } e_1 \in W. \text{ Then we can find scalars } a_i \text{ with } i \geq 2 \text{ such that } e_1 = \sum_{i \geq 2} a_i(e_i + e_i) = \sum_{i \geq 2} a_ie_i. \text{ Rearranging the terms we get } \sum_{i \geq 2} (a_i - 1)e_i + \sum_{i \geq 2} a_ie_i = 0 \text{ - as the } \{e_i\} \text{ are linearly independent, the } a_i \text{ must all be zero, contradicting that } a_i - 1 = 0. \text{ Thus } W \neq V. \text{ Now we show that } W^\perp = \{0\}; \text{ if } \langle \sigma, \mu \rangle = 0 \forall n \in N, \text{ then } \sum_{i \geq 2} (e_i + e_n)(k)\mu(k) = 0. \text{ i.e. } \sum_{k} e_1(k)\mu(k) = -\sum_{k} e_n(k)\mu(k). \text{ This holds for every } n \text{ exactly when } \mu(1) = -\mu(n) \forall n \in N. \text{ Such a } \mu \text{ is an element of } W \text{ iff } \mu(n) = 0 \forall n \in N \text{ (otherwise, the sequence would have infinitely many nonzero terms). Thus } W^\perp = \{0\} \text{ and so } (W^\perp)^\perp = \{0\}^\perp = V \neq W.

6.3.1 \text{ (a)T (b)F (c) T (d) F (f) T (g) T}

6.3.2(b) We want find \((y_1, y_2)\) such that \(z_1 - 2z_2 = g(z_1, z_2) = \langle (z_1, z_2), (y_1, y_2) \rangle = z_1g_1 + z_2g_2\). We see that \((1, -2)\) works.

6.3.3(a) We see that with respect to the standard basis, \([T] = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\), which is equal to its conjugate transpose. We have that \([T^*] = [T]^* = [T]\) and so the operator is self-adjoint.

6.3.3(b) With respect to the standard basis, \([T] = \begin{pmatrix} 2 & i \\ 1 - i & 0 \end{pmatrix}\) and so \([T^*] = [T]^* = \begin{pmatrix} 2 & 1 + i \\ -i & 0 \end{pmatrix}\). Thus, we see \(T(z_1, z_2) = (2z_1 + (1 + i)z_2, -iz_1)\)

6.3.6 Given \(U_1 = T + T^\ast\), we calculate \(U_1^\ast = (T + T^\ast)^\ast = T^\ast + T^{**} = T^\ast + T = T + T^\ast = U_1\)
\text{ Given } U_2 = TT^\ast \text{ we calculate } U_2^\ast = (TT^\ast)^\ast = (T^\ast)^*T^* = TT^* = U_2.

6.3.9 If \(V = WW^\perp\) and \(T\) is the projection on \(W\) along \(W^\perp\), we see that \(T|_W = Id\) and \(T|_{W^\perp} = 0\). So if we choose a basis for \(V\), \(\beta = \{v_1, \ldots, v_k, w_1, \ldots, w_m\}\)
where \{v_i\} is a basis for \(W\) and \{w_i\} is a basis for \(W^\perp\). Then we see \([T] = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}\), which is self-adjoint. Therefore, by Thm 6.10, \(T\) is self-adjoint.

6.3.10 (\(\iff\)) This follows from taking \(x = y\).
(\(\Rightarrow\)) We use the polar identity:

\[
\langle x, y \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \|x + i^k y\|^2
\]

(5)

\[
= \frac{1}{4} \sum_{k=1}^{4} i^k \|T(x) + i^k T(y)\|^2
\]

(6)

\[
= \frac{1}{4} \sum_{k=1}^{4} i^k \langle T(x) + i^k T(y), T(x) + i^k T(y) \rangle
\]

(7)

\[
= \frac{1}{4} \sum_{k=1}^{4} i^k (\|T(x)\|^2 + \|T(y)\|^2 + 2 \Re \langle T(x), i^k T(y) \rangle)
\]

(8)

\[
= \frac{1}{4} \sum_{k=1}^{4} 2 \Re \langle T(x), i^k T(y) \rangle
\]

(9)

\[
= \langle T(x), T(y) \rangle
\]

(10)

Where we go from (5) to (6) using the hypothesis, and (6) to (7) by the definition of the norm. If you’re having trouble going from (9) to (10), write \(\langle T(x), T(y) \rangle = a + ib\) (so \(\Re \langle T(x), T(y) \rangle = a\) and \(\Im \langle T(x), T(y) \rangle = b\)) and write out each of the terms of the sum in terms of \(a\) and \(b\).

6.3.12(a) Pick \(x \in R(T^* +)\), then \(\forall T^* y \in R(T^*)\) we have \(0 = \langle x, T^* y \rangle = \langle Tx, y \rangle\).
Since this holds for all \(y\), we can take \(y = Tx\) to get \(\langle Tx, Tx \rangle = 0\). Thus \(Tx = 0\) and so \(x \in N(T)\).
Now, pick \(x \in N(T)\). Then \(0 = \langle Tx, y \rangle = \langle x, T^* y \rangle \forall y\), i.e. \(x \in R(T^*)^\perp\).

6.3.12(b) This follows immediately from the proposition that states: if \(W\) is a finite dimensional subspace of an inner product space, then \((W^\perp)^\perp = W\). Proof:
pick \(x \in W\), then it’s immediate that \(\langle x, y \rangle = 0 \forall y \in W^\perp\), and thus \(x \in (W^\perp)^\perp\).
Now, suppose \(x \in (W^\perp)^\perp\).
Now if \(x \notin W, 6.2.6\) gives the existence of a \(y \in W^\perp\) such that \(\langle x, y \rangle \neq 0\) – i.e. \(x \notin (W^\perp)^\perp\), a contradiction.

6.3.13(a) That \(N(T) \subseteq N(T^* T)\), follows from the fact that \(T(0) = 0\). We now prove \(N(T^* T) \subseteq N(T)\): pick \(x \in N(T^* T)\), so \(Tx \in N(T^*) = R(T)^\perp\).
Thus \(\forall v \in V\) we have \(\langle Tx, Tv \rangle = 0\). Taking \(v = x\) we get \(\langle Tx, Tx \rangle = 0\) – i.e. \(Tx = 0\).
Now since the nullspaces are equal, it follows that their nullities are equal.
Thus, by the dimension theorem, their Ranks are equal.

6.3.13(b) Note that \(R(T^*) = N(T)^\perp\) and so \(Rk(T^*) = \dim(N(T)^\perp) = n-\dim N(T) = Rk(T)\).
The second equality follows from Theorem 6.7(c), and the third equality is the dimension theorem. Thus we have that \(Rk(TT^*) = Rk((T^*)^* T^*) = \frac{\mathbf{1}}{4} \sum_{k=1}^{4} i^k \|x + i^k y\|^2\)
\[ \text{Rk}(T^*) = \text{Rk}(T) = \text{Rk}(T^*T). \] The second and fourth equalities are from part (a).

6.3.13(c) This follows immediately from \( \text{Rk}(L_A) = \text{Rk}(A) \) and \( [L_A^*] = A^* \).