Math 110, Professor Ogus, Homework due 2/4

6.1.1 (a)T (b)T (c) F (d) F (e) T (f) F (g) F (h) T

3. Let $f(t) = t$ and $g(t) = e^t$.
$\langle f, g\rangle = \int_0^1 te^t dt = 1$
$\|f\| = (\int_0^1 t^2 dt)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$
$\|g\| = (\int_0^1 e^{2t} dt)^{\frac{1}{2}} = \sqrt{\frac{e^2-1}{2}}$
$\|f+g\| = (\int_0^1 (t+e^t)^2 dt)^{\frac{1}{2}} = \sqrt{\frac{11+3e^2}{6}}$
To check that the Cauchy-Schwarz inequality holds, one checks that
$$1 = |\langle f, g\rangle| \leq \|f\|\|g\| = \frac{\sqrt{3}}{3}\sqrt{\frac{11+3e^2}{6}}$$
To check the triangle inequality holds, one checks that
$$\sqrt{\frac{11+3e^2}{6}} = \|f+g\| \leq \|f\| + \|g\| = \frac{\sqrt{3}}{3} + \sqrt{\frac{e^2-1}{2}}$$

4b. The Frobenius inner product is given by $\langle A, B\rangle = tr(B^*A)$. Thus we have
$$\|A\|^2 = tr(A^*A) = tr(\begin{pmatrix} 1 & 3 \\ 2-i & -i \end{pmatrix}\begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}) = tr(\begin{pmatrix} 10 & * \\ * & 6 \end{pmatrix}) = 16$$
And so $\|A\| = 4$. Similarly,
$$\|B\|^2 = tr(B^*B) = tr(\begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix}\begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}) = tr(\begin{pmatrix} 3 & * \\ * & 1 \end{pmatrix}) = 4$$
And so $\|B\| = 2$. Lastly, $\langle A, B\rangle = tr(B^*A) = tr(\begin{pmatrix} 1-i & -i \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}) =$ $tr(\begin{pmatrix} 1-4i & * \\ * & -1 \end{pmatrix}) = -4i$.

8b. $\langle A, B\rangle = tr(A+B)$ is not positive definite. Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\langle A, A\rangle = 0$, but $A \neq 0$.

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$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle & (1) \\ &= 2\|x\|^2 + 2\|y\|^2 + \langle x, y\rangle + \langle y, x\rangle - \langle x, y\rangle - \langle y, x\rangle & (2) \\ &= 2\|x\|^2 + 2\|y\|^2 & (3) \end{aligned}$$

$x+y$ and $x-y$ can be viewed as the diagonals of the parallelogram spanned by x and y. The parallelogram law then says the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the sides of the parallelogram.

15a. We want to prove that $|\langle x,y\rangle| = \|x\|\|y\|$ if and only if $x = cy$ for some scalar c. First assume $x = cy$. Then $|\langle x,y\rangle| = |\langle cy,y\rangle| = |c||\langle y,y\rangle| = |c|\|y\|^2 = \|cy\|\|y\| = \|x\|\|y\|$ as desired. Now assume that $|\langle x,y\rangle| = \|x\|\|y\|$. If $y = 0$, then $c = 0$ works. Assume that $y \neq 0$. Let $a = \frac{\langle x,y\rangle}{\|y\|^2}$ and $z = x - ay$. First note that y and z are orthogonal: $\langle z,y\rangle = \langle x - ay, y\rangle = \langle x,y\rangle - a\langle y,y\rangle = \langle x,y\rangle - \langle x,y\rangle = 0$. Thus it follows that ay and z are orthogonal. We therefore have

$$\|x\|^2 = \|ay + z\|^2 = \|ay\|^2 + \|z\|^2 \tag{4}$$

, where the last equality holds by 6.1 problem 10. However, $|a| = |\frac{\langle x,y\rangle}{\|y\|^2}| = \frac{\|x\|\|y\|}{\|y\|^2} = \frac{\|x\|}{\|y\|}$ (the middle equality holding by hypothesis), and so $\|ay\|^2 = \|x\|^2$. The equation (*) becomes $\|x\|^2 = \|x\|^2 + \|z\|^2$, giving $\|z\|^2 = 0$. Because inner products are positive definite, this implies that $z = 0$, i.e. $x = ay$ as desired.

15b. We want to show that $\|x + y\| = \|x\| + \|y\|$ iff $x = cy$ for some scalar c.

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle && (5)\\
&= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle && (6)\\
&= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle x,\bar{y}\rangle && (7)\\
&= \|x\|^2 + \|y\|^2 + 2\Re\langle x,y\rangle && (8)\\
&\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| && (9)\\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| && (10)\\
&= (\|x\| + \|y\|)^2 && (11)
\end{aligned}$$

The two inequalities hold iff the Cauchy-Schwarz inequality is an equality, which in turn holds iff $x = cy$ (by part a).
We can generalize this result to the case of n vectors as follows: $\|x_1 + \ldots + x_n\| = \|x_1 + (x_2 \ldots + x_n)\| = \|x_1\| + \|x_2 \ldots + x_n\| = \ldots = \|x_1\| + \ldots + \|x_n\|$.

6.2.1 (a) F (b) T (c) T (d) F (e) T (f) F (g) T

2c. We apply Gram-Schmidt to the standard basis $\{1, x, x^2\}$ of $P_2(R)$ with respect to the inner product $\langle f,g\rangle = \int_0^1 fg dx$. In other words, we'll find an orthogonal set $S = \{v_0, v_1, v_2\}$ with the same span as $\{1, x, x^2\}$. First, $v_0 = 1$. Second, $v_1 = x - \langle x,1\rangle = x - \frac{1}{2}$ (as $\langle x,1\rangle = \int_0^1 xdx = \frac{1}{2}$). Lastly, to compute v_2, we will need the following computations: $\langle x^2, x - \frac{1}{2}\rangle = \int_0^1 x^2(x - \frac{1}{2})dx = \frac{1}{12}$ and $\|x - \frac{1}{2}\|^2 = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$. Now, $v_2 = x^2 - \frac{\langle x^2, x - \frac{1}{2}\rangle}{\|x - \frac{1}{2}\|^2}(x - \frac{1}{2}) - \langle x^2, 1\rangle = x^2 - (x - \frac{1}{2}) - \frac{1}{3} = x^2 - x + \frac{1}{6}$. Thus we have $S = \{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$. To normalize this basis, we compute the norms of the elements: $\|v_0\| = \|1\| = 1$, $\|v_1\| = \|x - \frac{1}{2}\| = \frac{1}{2\sqrt{3}}$, $\|v_2\| = (\int_0^1 (x^2 - x + \frac{1}{6})^2 dx)^{\frac{1}{2}} = \frac{1}{6\sqrt{5}}$ Thus our orthonormal basis is $\beta = \{1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\}$.

The Fourier coefficients of $h(x) = x + 1$ are: $\langle h, 1\rangle = \int_0^1 (x+1)dx = \frac{3}{2}$, $\langle h, x - \frac{1}{2}\rangle = \int_0^1 (x+1)(x-\frac{1}{2})dx = \frac{\sqrt{3}}{6}$, $\langle h, x^2 - x + \frac{1}{6}\rangle = \int_0^1 (x+1)(x^2 - x + \frac{1}{6})dx = 0$

4. To find the orthogonal complement to $S = \{(1, 0, i), (1, 2, 1)\}$, we find all $(x, y, z) \in C^3$ such that:

$$\langle x, y, z\rangle \cdot \langle 1, 0, i\rangle = x - zi = 0 \tag{12}$$
$$\langle x, y, z\rangle \cdot \langle 1, 2, 1\rangle = x + 2y + z = 0 \tag{13}$$

   We obtain $x = zi$ and $y = -\frac{1+i}{2}z$, giving $S^{\perp} = span\{(i, -\frac{1+i}{2}, 1)\}$

5. If $S = \{x_0\}$, where $x_0 \neq 0$ is a vector in R^3, $S^{\perp}$ is the plane through the origin containing vectors orthogonal to x_0. Given a linearly independent subset $S = \{x_1, x_2\} \subseteq R^3$, $S^{\perp}$ is the line through the origin containing vectors orthogonal to the plane determined by S.