Math 110, Professor Ogus, Homework due 3/21

(written by Janak Ramakrishnan)

5.2.18a. We show $TU = UT$ by considering action on elements of β. As β is a basis, this is sufficient. Let v be an arbitrary element of β. Let λ be the eigenvalue of T corresponding to v, and μ the eigenvalue of U corresponding to v. Then $TUv = T(\mu v) = \mu Tv = \mu\lambda v = \lambda\mu v$. $UTv = U(\lambda v) = \lambda Uv = \lambda\mu v$.

5.2.19. Let β be an ordered basis in which $[T]_\beta$ is diagonal. By 5.1.15(a), every $v \in \beta$ is an eigenvector of T^m for any $m > 0$, showing that $[T^m]_\beta$ is also diagonal.

5.2.20. We prove the forward direction first. Let β_i be a basis of W_i, for $1 \le i \le k$. It is clear that the β_i's are pairwise disjoint, since $W_j \cap \sum_{i \neq j} W_i = \{0\}$, so certainly $W_j \cap W_i = \{0\}$. Let $\alpha = \beta_1 \cup \ldots \cup \beta_k$. It is clear that $|\alpha| = \sum_{i=1}^k |\beta_i| = \sum_{i=1}^k \dim(W_i)$, and by Theorem 5.10, α is a basis for V, so $\dim(V) = |\alpha| = \sum_{i=1}^k \dim(W_i)$.

For the reverse direction, take the same setup of β_i's and α. Once again, it is clear that α spans V, and since $|\alpha| \le \sum_{i=1}^k |\beta_i| = \sum_{i=1}^k \dim(W_i) = \dim(V)$, α is a basis of V. Now by Theorem 5.10 again, $V = \bigoplus_{i=1}^k W_i$.

5.2.22. First we show that $\mathrm{Span}(\{x \in V \mid x \text{ is an eigenvector of } T\}) = \sum_{i=1}^k E_{\lambda_i}$. This is clear, since a vector is in the first space if and only if it is a linear combination of eigenvectors of T, which is precisely the set of vectors in the second space. Next we show that $E_{\lambda_j} \cap \sum_{i \neq j} E_{\lambda_i} = \{0\}$, for any j. Fix an arbitrary j, and let $v \in E_{\lambda_j} \cap \sum_{i \neq j} E_{\lambda_i}$. We will show $v = 0$. We can write $v = \sum_{i \neq j} x_i$, where each x_i is in E_{λ_i}. Then $0 = -v + \sum_{i \neq j} x_i$. If any of the vectors on the right side is nonzero, then we would have a linear combination of eigenvectors which was 0, which is impossible by Theorem 5.5. Thus, all the vectors must be 0, so $v = 0$.

5.4.1. (a) F, (b) T, (c) F, (d) F, (e) T, (f) T, (g) T.

5.4.2ace. (a) Yes – T can only decrease the degree of an element of V, and $P_2(R)$ is closed downwards with respect to degree. (c) Yes – $T(V) \subseteq W$, so $T(W) \subseteq W$. (e) No – $T\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, so $T(W) \not\subseteq W$.

5.4.4. Since W is a subspace, it is closed under linear combination. Since $g(T)(w)$, for $w \in W$, is a linear combination of vectors of the form $T^m(w)$ for $m \ge 0$, we need only show that $T^m(w) \in W$ for $m \ge 0$ and every $w \in W$. Show this by induction on m. For $m = 0$, $T^m(w) = w \in W$. For $m = k+1$, we have $T^m(w) = T^k(T(w))$. Since $T(w) \in W$, by induction we know that $T^k(T(w)) \in W$, and so we are done.

5.4.6bd. (b) $T(z) = 6x$, and $T^2(z) = 0$. Then $\{x^3, x\}$ is a basis. (d) $T(z) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, and $T^2(z) = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, so $\left\{z, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\right\}$ is a basis.

5.4.9bd. (b) We know that $T^2(z) = 0$. Thus $0x^3 + 0T(x^3) + T^2(x^3) = 0$, so $f(t) = (-1)^2(t^2) = t^2$. Computing using the determinant, we have $[T]_\beta = \begin{pmatrix} 0 & 0 \\ 6 & 0 \end{pmatrix}$, so $\det([T]_\beta) = t^2$. (d) $-2T(z)+T^2(z) = 0$, so $f(t) = t^2 - 2t$. Computing using the determinant, we have $[T]_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$, so $\det([T]_\beta) = t^2 - 2t$.

5.4.13. The forward direction is clear – w is a linear combination of vectors of the form $T^m(v)$, for $m \geq 0$. This linear combination gives g. The reverse is also clear – if w can be written as $g(T)(v)$, then it is a linear combination of powers of T.

5.4.18. (a) $f(t) = \det(A - tI)$. $a_0 = f(0) = \det(A - 0I) = \det(A)$, and $\det(A) \neq 0$ iff A is invertible. (b) We know that $f(A) = 0$. Then

$$\begin{aligned} (-1)^n A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n &= 0 \\ (-1)^n A^n + a_{n-1}A^{n-1} + \ldots + a_1 A &= -a_0 I_n. && \text{Distributing out an } A \text{ and dividing by } -a_0, \\ A((-1/a_0)(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I_n) &= I_n. && \text{By definition,} \\ A^{-1} = (-1/a_0)(-1)^n A^{n-1} + a_{n-1}A^{n-2} &+ \ldots + a_1 I_n \end{aligned}$$

5.4.20. In the reverse direction, the statement holds for any V, simply because $Tg(T) = g(T)T$ for any polynomial g. This is because of distributivity and associativity of linear operators, and commutativity of T with scalar multiples of I. In the forward direction, let v generate V. Then $V = \mathrm{Span}(\{v, Tv, T^2v, \ldots\})$. Since $Uv \in V$, we can write Uv as a finite linear combination of vectors in the form $T^m v$, $m \geq 0$ – say $Uv = a_0 v + a_1 Tv + \ldots + a_k T^k v$. Let $g(t) = a_0 + a_1 t + \ldots + a_k t^k$. We claim $U(w) = g(T)(w)$ for any $w \in V$. It suffices to check this for a spanning set, so we can take $w = T^m v$, for $m \geq 0$. Note that, since $UT = TU$, $UT^m = T^m U$, for $m \geq 0$, since we can repeatedly apply the identity $UT = TU$. Then we have $U(T^m v) = T^m(Uv) = T^m(g(T)v) = g(T)(T^m v)$, since $g(T)$ and T^m commute. Thus, $U(w) = g(T)(w)$ for all $w \in V$, and so $U = g(T)$.