Lectures on Logarithmic Algebraic Geometry—Errata

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Out of a stem that scored the hand
I wrung it in a weary land.

1 Misprints

Page 523, in Theorem 4.2.4, $\Omega^{\cdot}_{X/\mathbf{C}}$ should be $\Omega_{X/k}$.

Page 467, the reference to [16] should be to [13], and the item [16] should be omitted from the bibliography. (Thanks to J. Milne)

Page 294, in line 5 of Example 1.7.2, I_i shoud be I''_i, and in line 6, $I_i \otimes O_X$ should be $I''_i \to O_{X''}$. (Thanks to E. Goncharov)

Page 385, line 4 of Variant 2.2.3, T_1 should be T. (Thanks to E. Goncharov)

2 Correction to Proposition V.2.1.12

Helge Ruddat has informed me of an example due to Simon Felten [3, Example 7.5] that implies that statement (2) of Proposition V.2.1.12 and Corollary V.2.3.15 are not correct as stated. The point of the corollary was to show that, if X is a smooth log scheme over a regular ring R with trivial log structure, then our definition of the complex $\underline{\Omega}^{\cdot}_{X/R}$ agrees with the complex $j_*\Omega^{\cdot}_{\underline{U}/R}$, where $j\colon \underline{U} \to \underline{X}$ is the inclusion of the regular locus of X/R; the latter is the complex considered by Danilov in his pioneering work [2]. As it turns out this is true if R is flat over $\mathbf{Z}$, but not in general. It appears that each of the two constructions has its own advantages and disadvantages. For example, our construction commutes more often with base change (see Proposition V.2.3.26). On the other hand, our sheaves may not be reflexive, and the pairing $\underline{\Omega}^{i}_{X/R} \otimes \underline{\Omega}^{n-i}_{X\mathrm{R}} \to \underline{\Omega}^{n}_{X/R}$ (where $n := \dim X/R$) may not be perfect, in contrast to the sheaves considered by Danilov [2, 4.7]. Both constructions admit a Cartier isomorphism; see Proposition V.4.1.3 and [1].

Example 2.0.1. Let Q be the monoid given by generators a, b, c satisfying the relation $a + b = 2c$. This monoid is fine and saturated, and its facets F_1 and F_2 are the submonoids generated by a and b respectively. Thus $F_1^{\mathrm{gp}} \cap F_2^{\mathrm{gp}} = \{0\}$, but in $Q^{\mathrm{gp}} \otimes \mathbf{F}_2$, $a = b$ so $(F_1^{\mathrm{gp}} \otimes \mathbf{F}_2) \cap (F_2^{\mathrm{gp}} \otimes \mathbf{F}_2) = F_1^{\mathrm{gp}} \otimes \mathbf{F}_2$. Recall from Definition V.2.3.4 that $\underline{\Omega}^1_{Q/R}$ is the Q-graded submodule of $R[Q] \otimes Q^{\mathrm{gp}}$ which in degree q is $R \otimes \langle q\rangle^{\mathrm{gp}}$. On the other hand, as is explained in the course of the proof of Proposition V.2.3.13 and in [2, 4.3], $\Omega^1_{U/R}(U)$ is the Q-graded submodule of $R[Q] \otimes Q^{\mathrm{gp}}$ which in degree q is the intersection of the submodules $R \otimes F^{\mathrm{gp}}$ as F ranges over the facets of Q containing q. Thus if $R = \mathbf{F}_2$, we find that, in degree 0, $\underline{\Omega}^1_{Q/R,0} = 0$ while $\Omega^1_{U/R}(U) = F_1^{\mathrm{gp}} \otimes R = F_2^{\mathrm{gp}} \otimes R$.

As the paper [3] explains in Lemma 7.6 and Corollary 7.9, an "elementary exercise in Tor groups" shows that the difficulty disappears in sufficiently large characteristics, depending on the log scheme in question. As a penance, we carry out this exercise with an attempt to make this dependence more explicit.

Lemma 2.0.2. *Let E be a finitely generated free abelian group, let T be a finite set of subgroups of E, and for each subset S of T, let $H_S := \cap\{H : H \in S\}$ and $T_S := \{H_S \cap H : H \in T\}$. Finally, let N be the product of all the primes dividing the orders of the torsion subgroups of the groups $H_S/(H_1 + H_2)$, as (H_1, H_2) ranges over the set of pairs of elements of T_S, for all $S \subseteq T$. Then if R is a ring such that $\mathrm{Tor}_1^{\mathbf{Z}}(R, \mathbf{Z}/N\mathbf{Z}) = 0$, the following conclusions hold.*

1. *For every $H \in T$, the natural map $R \otimes H \to R \otimes E$ is injective.*

2. *For every $S \subseteq T$, the natural map:*

$$a_S : R \otimes H_S \to (R \otimes H)_S := \cap\{R \otimes H : H \in S\} \subseteq R \otimes E$$

is an isomorphism.

3. *Suppose also that each E/H for $H \in S$ has rank one. Then for every i and every $H \in S$, the natural map $R \otimes \Lambda^i H \to R \otimes \Lambda^i E$ is injective, and for every $S \subseteq T$, the natural map*

$$a_{S,i} \colon R \otimes \Lambda^i H_S \to \cap\{R \otimes \Lambda^i H : H \in S\}$$

is an isomorphism.

Proof. Our hypothesis implies that $\mathrm{Tor}_1^{\mathbf{Z}}(R, \mathbf{Z}/p\mathbf{Z}) = 0$ for every p dividing N and then that $\mathrm{Tor}_1^{\mathbf{Z}}(R, H_S/(H_1 + H_2)) = 0$ for every $S \subseteq T$ and every $(H_1, H_2) \in T_S \times T_S$. Observe that for every $S \subseteq T$, the family T_S of subgroups of $E_S := H_S$ enjoys the same property as the family T of subgroups of E and that $H_\emptyset = E$ and $T_\emptyset = T$. In particular, $\mathrm{Tor}_1^{\mathbf{Z}}(E/H, R) = 0$ for every $H \in T$, so the map $R \otimes H \to R \otimes E$ is injective. This proves (1), and in particular, it makes sense to write $\cap\{R \otimes H : H \in S\} \subseteq R \otimes E$. The natural map $R \otimes H_S \to R \otimes H$ factors through $\cap\{R \otimes H : H \in S\}$, inducing the "natural map" a_S in the statement. We shall prove that this map is an isomorphism by induction on the cardinality of S. If $|S| = 1$, the result is clear, since we already know that each $R \otimes H \to R \otimes E$ is injective. If $|S| = 2$ and $S = \{H_1, H_2\}$, we consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \otimes (H_1 \cap H_2) \arrow[r] \arrow[d, "a_S"'] & R \otimes (H_1 \oplus H_2) \arrow[r] \arrow[d, "\cong"'] & R \otimes (H_1 + H_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (R \otimes H_1) \cap (R \otimes H_2) \arrow[r] & R \otimes H_1 \oplus R \otimes H_2 \arrow[r] & R \otimes H_1 + R \otimes H_2 \arrow[r] & 0
\end{tikzcd}$$

The bottom sequence is exact by inspection, the top sequence is exact because $H_1 + H_2$ is a free abelian group, and it follows that a_S is injective. Then the snake lemma implies that the cokernel of a_S is isomorphic to the kernel of the right vertical arrow, which is $\mathrm{Tor}_1^{\mathbf{Z}}(R, E/(H_1 + H_2)) = 0$.

For the induction step, choose some element E' of S and let $S' := \{H' := H \cap E' : H \in S \setminus \{E'\}\}$. Consider the diagram:

$$\begin{tikzcd}
R \otimes H'_{S'} \arrow[r, "a_{S'}"] \arrow[d] & \cap\{(R \otimes H' : H' \in S'\} \arrow[d] \\
R \otimes H_S \arrow[r, "a_S"] & \cap\{R \otimes H \cap R \otimes E' : H \in S \setminus \{E'\}\}
\end{tikzcd}$$

Here the intersection in the top right corner is taken in $R \otimes E'$ and the intersection on the bottom right in $R \otimes E$; since $R \otimes E' \subseteq R \otimes E$ this distinction is vacuous. Since the collection $T' := \{H \cap E' : H \in T\}$ of subgroups of E' satisfies the hypotheses of the lemma and $|S'| < |S|$, the induction hypothesis implies that $a'_{S'}$ is an isomorphism. By the case $n = 2$, the right vertical arrow is an isomorphism, and the left vertical arrow is trivially an isomorphism—in fact an equality. It follows that the bottom horizontal arrow is an isomorphism, proving conclusion (2) of the lemma.

We prove statement (3) by induction on i. The case $i = 1$ is covered by statement (2). Let us explain the induction step when $S = \{H_1, H_2\}$; in what follows we write $H_{1,2}$ for $H_1 \cap H_2$. Since $H_2 \to E \to E/H_2$ is a short exact sequence of free abelian groups and E/H_2 has rank one, the Koszul construction gives a short exact sequence:

$$0 \to \Lambda^i H_2 \to \Lambda^i E \to \Lambda^{i-1} H_2 \otimes E/H_2 \to 0,$$

and this sequence remains exact after tensoring with R. We get a similar exact sequence from the corank direct summand $H_{1,2}$ of H_2. Thus each of the maps

$$R \otimes \Lambda^i(H_{1,2}) \to R \otimes \Lambda^i H_2 \text{ and } R \otimes \Lambda^i H_2 \to R \otimes \Lambda^i E$$

is injective, and hence so is their composition. Thus the rows in the diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \otimes \Lambda^i H_{1,2} \arrow[r] \arrow[d] & R \otimes \Lambda^i H_1 \arrow[r] \arrow[d] & R \otimes \Lambda^{i-1} H_{1,2} \otimes H_1/H_{1,2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \otimes \Lambda^i H_2 \arrow[r] & R \otimes \Lambda^i E \arrow[r] & R \otimes \Lambda^{i-1} H_2 \otimes E/H_2 \arrow[r] & 0
\end{tikzcd}$$

are exact, and it will suffice to show that the vertical arrow is injective. This arrow factors as a composition of arrows:

$$\begin{aligned}
R \otimes \Lambda^{i-1}(H_{1,2}) \otimes H_1/H_{1,2} &\to R \otimes \Lambda^{i-1} H_2 \otimes H_1/H_{1,2} \\
R \otimes \Lambda^{i-1} H_2 \otimes H_1/H_{1,2} &\to R \otimes \Lambda^{i-1} H_2 \otimes E/H_2
\end{aligned}$$

each of which is easily seen to be injective. (The latter because of the vanishing of $\mathrm{Tor}_1^{\mathbf{Z}}(R \otimes \Lambda^{i-1} E \otimes E/(H_1 + H_2)$.) $\square$

We can now state a corrected version of statement (2) of Proposition V.2.3.12 and its Corollary V.2.3.15. We restrict here to the case in which $P = 0$.

Proposition 2.0.3. *Let Q be a fine saturated monoid, let T denote the set of all subgroups of Q^{gp} of the form G^{gp} as G ranges over the facets of Q, and let N_Q be the natural number corresponding to this (finite) set of subgroups of Q^{gp} as described in Lemma 2.0.2. Suppose that F is a face of Q, that U is the inverse image in $\underline{\mathsf{A}}_{\mathsf{Q}}$ of an open subset of $\mathrm{Spec}(Q)$ containing all the height one primes of Q, and that R is a ring such that $\mathrm{Tor}_1^{\mathbf{Z}}(R, \mathbf{Z}/N_Q\mathbf{Z}) = 0$. Then the natural map*

$$\underline{\Omega}^{\cdot}_{Q/R}(F) \to \Gamma(U, \underline{\Omega}^{\cdot}_{Q/R}(F))$$

is an isomorphism.

Proof. The proof proceeds as before, with the following argument replacing the original version, starting with the last paragraph on page 487:

Each G is a facet of Q, so $\langle F, G, q\rangle$ is G if G contains F and q and is Q otherwise. Thus Lemma 2.3.13 implies that

$$\Gamma(U, L^0(F)\tilde{\Omega}^i_{Q/R})_q \subseteq \bigcap_G \Omega^i_{Q/R,G,q} = \bigcap_{G\supseteq\langle F,q\rangle} R\otimes \Lambda^i G^{\mathrm{gp}}.$$

As we saw in Corollary I.2.3.14, the intersection of the set of all G^{gp} such that G contains $\langle F, q\rangle$ is just $\langle F, q\rangle^{\mathrm{gp}}$. Applying statement (3) of Lemma 2.0.2, we see that

$$\bigcap_{G\supseteq\langle F,q\rangle} R\otimes \Lambda^i G^{\mathrm{gp}} = R\otimes \bigcap_{G\supseteq\langle F,q\rangle} \Lambda^i G^{\mathrm{gp}} = R\otimes \langle F, q\rangle^{\mathrm{gp}} = \underline{\Omega}^i_{Q/R}(F). \qquad \square$$

Corollary 2.0.4. *Let Q be a fine saturated monoid, and let U be the inverse image in* $\operatorname{Spec} R[Q]$ *of the set of $\mathfrak{p} \in \operatorname{Spec} Q$ such that* $\operatorname{ht}\mathfrak{p} \leq 1$. *Assume that the order of the torsion subgroup of Q^{gp} is invertible in R and that* $\operatorname{Tor}_1^{\mathbf{Z}}(R, \mathbf{Z}/N_Q\mathbf{Z}) = 0$, *where N_Q as as in Proposition 2.0.3. Then U is smooth over* $\operatorname{Spec} R$, *and the natural map*

$$\underline{\Omega}^{\cdot}_{Q/R} \cong \Gamma(\underline{U}, \Omega^{\cdot}_{\underline{U}/R})$$

is an isomorphism.

Remark 2.0.5. For example, if Q is a free monoid, then $N_Q = 1$, If Q is the monoid of example 2.0.1, then $N_Q = 2$, and if Q is the monoid given by a, b, c, d with $a+b = c+d$ we again $N_Q = 1$. Note also If Q is a toric monoid of rank n and q is an element of $Q \setminus I_Q$, then q is contained in some facet G, hence $\cap\{R\otimes G^{\mathrm{gp}} : q \in G\}$ has rank less than n, and hence $\cap\Lambda^n R\otimes G^{\mathrm{gp}} = 0$. It follows that $\underline{\Omega}^n_{Q/R} = \Gamma(U, \tilde{\Omega}^n_{U/R})$.

3 Corrections to section II.2.5

Ofer Gabber has observed that section II,2.5 contains a plethora of serious errors, triggered by two key issues. The first of these is that condition (2) in the Definition 2.5.1 is not stable under pull-back, and in fact it is not true that every fine log scheme admits a stratification satisfying these conditions. The second problem is that the cospecialization maps on page 263 are not well-defined in the generality stated. It is my view that these issues warrant considerable further study. At present I am only able to address these difficulties in a superficial manner.

For the sake of clarity, it seems best to replace section II,2.5 with the following revised version. I am deeply grateful to Gabber for his suggestions.

2.5 Constructibility and coherence

It is possible to give a fairly explicit description of what it means for a sheaf of integral monoids on a topological space to be coherent.

Definition 2.5.1. *Let $\mathcal{E}$ be a sheaf of sets on a topological space X. A trivializing partition for $\mathcal{E}$ is a locally finite partition Π of X into locally closed subsets such that the restriction of $\mathcal{E}$ to each $S \in \Sigma$ is constant. We say that a sheaf $\mathcal{E}$ on X is quasi-constructible if X has a trivializing partition for $\mathcal{E}$.*

Remark 2.5.2. Recall that a *Kolmogoroff space* is a topological space X such that given any two distinct points x and y of X, either x does not belong to the closure of y or y does not belong to the closure of x. For example, if Q is a monoid, $\operatorname{spec}(Q)$ is a Kolmogoroff space; it is finite if Q is finitely generated. Every point of a finite Kolmogoroff space is locally closed, and hence every sheaf on such a space is quasi-constructible. Furthermore, if $f: X \to Y$ is a continuous map and Σ is a trivializing partition for $\mathcal{E}$ on Y, then the family of nonempty members of $f^{-1}(\Sigma)$ is a trivializing partition for $f^{-1}(\mathcal{E})$ on X. It follows that every coherent sheaf of monoids admits a trivializing partition.

Remark 2.5.3. It is often useful for a partition Π to satisfy additional conditions that one finds in the literature concerning stratifications of topological space. Among these are the following.

1. Each element of Π is connected.
2. Each point of X admits a system of open neighborhoods whose intersection with every $S \in \Pi$ is either connected or empty.
3. Each element of Π is irreducible.
4. If T and S are elements of Π and $T \cap \overline{S} \neq \emptyset$, then $T \subseteq \overline{S}$ (the *frontier* condition).

Note that condition (3), which is mostly relevant for noetherian spaces, implies conditions (1) and (2). If Π is a trivializing parition for a sheaf on X, one can hope to find a refinement of Π which satisfies some of these additional conditions. For example, if X and the sets in Π admit compatible triangulizations (in a suitable sense which we shall not detail here), one should be able to do this for conditions (1), (2), and (4). If X is noetherian, one can achieve all four conditions, as we shall see in Proposition 2.5.5.

The following lemma may clarify the meaning of the frontier condition (4).

Lemma 2.5.4. *Let X be a topological space, let Π be a locally finite partition of X, and let Σ be the set of subsets of X that can be written as a union of elements of Π. Then the following conditions are equivalent.*

1. *The closure of every element of Σ also belongs to Σ.*
2. *The closure of every element of Π belongs to Σ.*
3. *The partition Π satisfies the frontier condition: whenever S and T are elements of Π and $T \cap \overline{S} \neq \emptyset$, then $T \subseteq \overline{S}$.*

Proof. It is obvious that (1) implies (2). Because Π is a partition of X, the set Σ is closed under intersections, complements, and unions. Suppose that (2) holds and that S and T are elements of Π. Then $T \cap \overline{S}$ belongs to Σ. Since T belongs to the partition

Π, any subset of T which belongs to Σ is either empty or all of T. Thus condition (2) implies condition (3). Suppose that Π satisfies (3) and that $S \in \Sigma$. Let t be a point of $\overline{S}$ and let T be the element of Π containing t. Choose a neighborhood U of t which meets only finitely many elements of Π. Since $S \in \Sigma$, we can write $S \cap U = S'_1 \cup S'_2 \cdots \cup S'_n$ where $S_1 \cdots S_n \in \Pi$ and $S'_i := S_i \cap U$. Then $\overline{S} \cap U = \overline{S \cap U} = \overline{S'_1} \cup \overline{S'_2} \cup \cdots \cup \overline{S}'_n$. Since $t \in T \cap \overline{S} \cap U$, it follows that $T \cap \overline{S}'_i \neq \emptyset$ for some i, and then $T \subset \overline{S'_i} \subseteq \overline{S}$. We have shown that every element of $\overline{S}$ is contained in an element T of Π which is contained in $\overline{S}$, and hence $\overline{S}$ is a union of elements of Π. $\square$

Proposition 2.5.5. [1] *Let X be a noetherian topological space.*

1. *If Σ is a finite cover of X by locally closed subsets, then X has a finite partition Π satisfying conditions (1)–(4) of Remark 2.5.3 such that every element of Σ is a union of elements of Π.*

2. *Every finite partition of X by locally closed subsets admits a finite refinement which satisfies the conditions (1)–(4) of Remark 2.5.3.*

Proof. We shall need the following lemma, of which part (3) is key.

Lemma 2.5.6. *Let X be a noetherian topological space.*

1. *X admits a finite partition into irreducible locally closed subspaces.*

2. *Every finite partition of X by locally closed subsets admits a finite refinement consisting of irreducible locally closed subsets.*

3. *Every finite collection $\mathcal{F}$ of irreducible closed subsets of X is contained in a finite collection $\hat{\mathcal{F}}$ of irreducible closed sets which contains the irreducible components of the intersection of any two elements of $\hat{\mathcal{F}}$.*

Proof. Statement (1) is trivial if X is irreducible. Since X is noetherian, it is the union of finitely many irreducible components $X_1, \dots, X_n$, and we can argue by induction on n. The space $X' := X_2 \cup \cdots \cup X_n$ has $n-1$ irreducible components, and the locally closed subset $X'_1 := X_1 \setminus X'$ is still irreducible. Since X' and X'_1 are disjoint and the lemma is true for each of them, it is also true for X. It is clear that statement (1) implies statement (2). If $\mathcal{F}$ is a finite family of irreducible closed subsets of X, let $\mathcal{F}'$ be the set of irreducible components of intersections of pairs of elements of $\mathcal{F}$, and consider the sequence $\mathcal{F}_0 := \emptyset, \mathcal{F}_1 := \mathcal{F}$ and $\mathcal{F}_{n+1} := \mathcal{F}'_n$ for $n > 0$. Any $Z \in \mathcal{F}_{n+1} \setminus \mathcal{F}_n$ for $n0$ is an irreducible component of an intersection of sets A, B in $\mathcal{F}_n$, neither of which contains the other and at least one of which, say A, does not lie in $\mathcal{F}_{n-1}$, so $Z \subsetneq A \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$. Then $Z_n := \cup\{Z \in \mathcal{F}_{n+1} \setminus \mathcal{F}_n\}$ is a closed subset of X, and, since $Z \subsetneq A \subseteq Z_{n-1}$, in fact $Z_n \subseteq Z_{n-1}$. Since A is irreducible and $Z \subsetneq A$, in fact Z is nowhere dense in A and hence also in Z_{n-1}. Since Z_n is a finite union of all such Z, it too is nowhere dense in Z_{n-1}. Since X is noetherian, the descending chain $Z_0, Z_1, \dots Z_n$ must terminate. Furthermore, Z_{n+1} is nowhere dense in Z_n, so the terminal element is empty. Then we can take $\hat{F} = \mathcal{F}_n$ for n sufficiently large to satisfy the requirements of statement (3). $\square$

[1] O. Gabber provided key help with the proof of this proposition. In fact his proof shows that "locally closed " can be replaced by "constructible," "noetherian" by "locally noetherian," and "finite" by "locally finite."

Statement (2) of the proposition is special case of statement (1). Let Σ be a finite cover of X by locally closed subsets. Replacing Σ by the family of irreducible components of its members, we may assume without loss of generality that each element of Σ is irreducible. Let $\mathcal{F}$ be the family of all irreducible components of sets of the form $\overline{A}$ or $\overline{A} \setminus A$ for $A \in \Sigma$, and let $\hat{\mathcal{F}}$ be a corresponding family as in statement (3). If $S \in \hat{\mathcal{F}}$, let

$$\hat{S} := S \setminus \cup\{T \in \hat{\mathcal{F}} : T \subsetneq S\} = S \setminus \cup\{S' \in \hat{\mathcal{F}} : S \not\subseteq S'\}.$$

The latter equality holds because if S and S' are distinct elements of $\mathcal{F}$, the irreducible components of $S \cap S'$ also belong to $\mathcal{F}$. Each of these sets $\hat{S}$ is nonempty and irreducible, and $\hat{S} \cap \hat{S}' = \emptyset$ if $S \neq S'$. If $x \in X$, let $\hat{\mathcal{F}}_x$ be the set of all $S \in \hat{\mathcal{F}}$ containing x. This set is not empty because Σ, and hence $\mathcal{F}$ and $\hat{\mathcal{F}}$, cover X. Since X is noetherian, the set $\hat{\mathcal{F}}_x$ has a minimal element S, and then $x \in \hat{S}$. This shows that the set Π of all sets of the form $\hat{S} : S \in \hat{\mathcal{F}}$ is a partition of X. Moreover, every element T of $\hat{F}_x$ must contain S, since otherwise some irreducible component of $T \cap S$ will be contain x, contradicting the minimality of S. This shows that every element of $\hat{\mathcal{F}}$ is a union of elements of Π. Note that if $P \in \Pi$, say $P = \hat{S}$ with $S \in \hat{\mathcal{F}}$, then $S = \overline{P}$, and consequently $\overline{P}$ is a union of elements of Π. Thus the partition Π satisfies the frontier condition. Suppose that $a \in A \in \Sigma$ and let P be the element of Π containing a. Since $\overline{A} \in \hat{\mathcal{F}}$, in fact $P \subseteq \overline{A}$. Since $\hat{F}$ contains all the irreducible components of $\overline{A} \setminus A$, it is also true that $\overline{A} \setminus A$ is a union of elements of Π, so $P \subset A$. This shows that A is a union of elements of Π and completes the proof of the proposition. $\square$

The frontier condition (4) is stable under restriction to open subsets, but not under general pullbacks. As a simple example let $X := \operatorname{Spec} k[x, y]$, let $\Pi := \{D(x), V(x)\}$, and let $i \colon X' \to X$ be the inclusion of the subscheme $V(xy)$. The partition Π satisfies condition (2), but its pullback Π' to X' is $\{V(y) \cap D(x), V(x)\}$ which does not. This can be remedied by replacing Π' by a refinement, e.g $\{V(y) \cap D(x), V(x) \cap D(y), V(x, y)\}$. However, if there are infinitely many irreducible components, such a locally finite refinement might not exist, as an example due to Gabber shows.

Example 2.5.7. Let $R := k[x_1, x_2, x_3, \ldots]/(x_1x_2, x_2x_3, x_3x_4, \ldots)$, let $X := \operatorname{Spec} R$, and let $\mathcal{M} \to \mathcal{O}_X$ be the log structure associated to the prelog structure $\mathbf{N} \to \mathcal{O}_X$ taking 1 to x_1. Then $\overline{\mathcal{M}}$ is the constant sheaf $\mathbf{N}$ on the closed subscheme $V(x_1)$ and is the constant sheaf 0 on its complement $D(x_1)$. The partition $\{V(x_1), D(x_1)\}$ does not satisfy condition (2), because $V(x_1) \cap \overline{D(x_1)} \neq \emptyset$ and $V(x_1) \not\subseteq \overline{D(x_1)} = V(x_2)$. In fact there is no locally finite partition of X into locally closed sets which satisfies condition (2) on which $\overline{\mathcal{M}}$ is constant. Suppose to the contrary that Π is such a partition, and let Σ denote the set of subsets of X which can be written as unions of elements of Π. Since $\overline{\mathcal{M}}$ is constant on the elements of Π, the sets $D(x_1)$ and $V(x_1)$ necessarily belong to Σ. Condition (2) implies that the closure, interior, and boundary of $D(x_1)$ also belong to Σ. Since $\partial D(x_1) = V(x_1, x_2)$, it also follows that $V(x_1) \setminus V(x_1, x_2) = V(x_1) \cap D(x_2)$ also belongs to Σ. Working within $V(x_1) \cong \operatorname{Spec} k[x_2, x_3, x_4, \ldots]/(x_2x_3, x_3x_4, \ldots)$, we see that the argument shows that $V(x_1, x_2, x_3)$ belongs to Σ, and then that $V(x_1, x_2, x_3, \ldots, x_n)$ belongs to Σ for every n. Since Π is locally finite, so is Σ, (i.e., every point has a neighborhood U such that $\{U \cap S : S \in \Sigma\}$ is finite), a contradiction.

Example 2.5.8. Let Q be a toric monoid and let $X := \underline{\mathsf{A}}_{\mathsf{Q}}(\mathbf{C})$, endowed with the strong topology. We claim that the partition of X into the orbits of $\underline{\mathsf{A}}^*_{\mathsf{Q}}$, described in Proposition I.3.3.4, satisfies conditions (1)–(2) and (4) of Remark 2.5.3. To verify the frontier condition, suppose that F and G are faces of Q with complementary prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ and that the closure of $\underline{\mathsf{A}}^*_{\mathsf{F}}$ meets $\underline{\mathsf{A}}^*_{\mathsf{G}}$. The closure of $\underline{\mathsf{A}}^*_{\mathsf{F}}$ is the closed subscheme $\underline{\mathsf{A}}_{\mathsf{F}}$ of $\underline{\mathsf{A}}_{\mathsf{Q}}$ defined by the ideal $\mathbf{C}[\mathfrak{p}]$ and $\underline{\mathsf{A}}^*_{\mathsf{G}}$ is the locally closed subscheme obtained by intersecting $\underline{\mathsf{A}}_{\mathsf{Q_G}}$ with the closed subscheme defined by the ideal $\mathbf{C}[\mathfrak{q}]$. Thus, if $\overline{\underline{\mathsf{A}}^*_{\mathsf{F}}} \cap \underline{\mathsf{A}}^*_{\mathsf{G}}$ is not empty, so is $\mathrm{Spec}(\mathbf{C}[Q]/\mathbf{C}[\mathfrak{p}])_G)$. This implies that $G \cap \mathfrak{p} = \emptyset$, i.e., that $G \subseteq F$, and hence that $\underline{\mathsf{A}}^*_{\mathsf{G}} \subseteq \underline{\mathsf{A}}_{\mathsf{F}}$, as required.

To verify conditions (1) and (2) notice that, since Q is toric, F^{gp} is torsion free for every face F of Q, so the stratum $\underline{\mathsf{A}}^*_{\mathsf{F}}$ is connected. Now let x be a point of $\underline{\mathsf{A}}_{\mathsf{Q}}$; we claim that x admits a basis of open neighborhoods as in (2). Let $G := \{q \in Q : e^q(x) \neq 0\}$. Then $\underline{\mathsf{A}}_{\mathsf{Q_G}}$ is an open neighborhood of x, so we may replace Q by Q_G and thus reduce to the case when $G = Q^*$. Since Q is saturated, we can write $Q \cong \overline{Q} \oplus Q^*$ and so $\underline{\mathsf{A}}_{\mathsf{Q}} \cong \underline{\mathsf{A}}_{\overline{\mathsf{Q}}} \times \underline{\mathsf{A}}_{\mathsf{Q^*}}$. Then $F \mapsto \overline{F} \oplus Q^*$ defines a bijection between the faces of Q and of $\overline{Q} \oplus Q^*$, and hence a corresponding bijection between the strata of $\underline{\mathsf{A}}_{\mathsf{Q}}$ and of $\underline{\mathsf{A}}_{\overline{\mathsf{Q}}} \times \underline{\mathsf{A}}_{\mathsf{Q^*}}$. Since the analytic space attached to $\underline{\mathsf{A}}_{\mathsf{Q^*}}$ is locally connected, we are reduced to the case in which Q is sharp and x is the vertex of $\underline{\mathsf{A}}_{\mathsf{Q}}$.

Let us now use the notation of §1.10, and in particular we let $\mathsf{C_Q}$ be the real cone spanned by Q. This space also admits a stratification by the interiors $\mathsf{C}^{\mathsf{o}}_{\mathsf{F}}$ of the cones of the faces of Q. Let $h\colon Q \to \mathbf{N}$ be a local homomorphism. The sets $B_h(r) := \{c \in \mathsf{C_Q} : h(c) < r\}$ form a basis of open neighborhoods of the vertex $0 \in \mathsf{C_Q}$. If F is a face of Q, let h_F be the composition of the projection $Q \to Q/F$ with a local homomorphism $Q/F \to \mathbf{N}$. Then $h_F^{-1}(0) = \mathsf{C_F} \subseteq \mathsf{C_Q}$, and

$$B_h(r) \cap \mathsf{C}^{\mathsf{o}}_{\mathsf{F}} = \{c \in \mathsf{C_Q} : 0 < h(c) < r\} \cap \{c : h_F(c) = 0\}$$

is convex, hence connected. This shows that the vertex admits a basis of open neighborhoods whose intersection with each of these strata is connected. Now Theorem 1.10.2 shows that the moment map associated to any set of generators of Q defines a homeomorphism from $\mathsf{R_Q}$ to $\mathsf{C_Q}$ which is compatible with the stratification by faces. Thus the vertex v of $\mathsf{R_Q}$ also has such basis $\mathcal{B}$ of open neighborhoods.

Now consider the commutative diagram:

$$\begin{tikzcd}
\mathsf{T_Q} \times \mathsf{R_Q} \arrow[d, "m"'] \arrow[dr, "pr"] & \\
\mathsf{X_Q} \arrow[r, "a"] & \mathsf{R_Q},
\end{tikzcd}$$

where m is multiplication and a is the absolute value function. The family of sets $a^{-1}(V)$ for $V \in \mathcal{B}$ forms an open neighborhood basis for the vertex x of $\mathsf{X_Q}$. If F is a face of Q, then $pr^{-1}(\mathsf{R}^*_{\mathsf{F}} \cap V) = \mathsf{T_Q} \times (\mathsf{R}^*_{\mathsf{F}} \cap V)$ is connected, and hence its image under m is also connected. Since the diagram commutes and m is surjective, this image is $a^{-1}(V \cap \mathsf{R}^*_{\mathsf{F}}) = a^{-1}(V) \cap \underline{\mathsf{A}}^*_{\mathsf{F}}$.

If X is a sober topological space [31, (0.2.1.1)], every irreducible closed set S contains a unique generic point σ. If $\mathcal{E}$ is a sheaf on X and $s \in S$, there is a natural cospecialization map

$$\mathrm{cosp}_{s,\sigma} \colon \mathcal{E}_s \to \mathcal{E}_\sigma,$$

which is an isomorphism if $\mathcal{E}$ is constant on S. Thus in this case we can identify $\Gamma(S, \mathcal{E}_{|_S})$ with $\mathcal{E}_s$ for every $s \in S$. Since we also want to work in the complex analytic context, we shall explain a different point of view.

Suppose $\mathcal{E}$ is a sheaf on a topological space X and that Π is a trivialization partition for $\mathcal{E}$ satisfying conditions (1), (2), and (4) of Remark 2.5.3. Since each $S \in \Pi$ is connected and since $\mathcal{E}_{|_S}$ is constant, the natural map

$$\mathcal{E}_S := \Gamma(S, \mathcal{E}_{|_S}) \to \mathcal{E}_s$$

is an isomorphism. If t belongs to the closure of S, and V is a neighborhood of t, then $S \cap V$ is not empty, and if this set is connected, we have a natural map

$$\mathcal{E}(V) \to \mathcal{E}_{|_S}(V \cap S) \cong \mathcal{E}_S.$$

Since Π satisfies condition (2) of Remark 2.5.3, the point t has a basis of neighborhoods with this property. Taking the limit over all such V, we find the *cospecialization map*,:

$$\mathrm{cosp}_{t,S} \colon \mathcal{E}_t \to \mathcal{E}_S.$$

If T is the stratum containing t, the map $\mathcal{E}_T \to \mathcal{E}_t$ is also an isomorphism, so we find by composition with its inverse a map:

$$\mathrm{cosp}_{T,S} \colon \mathcal{E}_T \to \mathcal{E}_S.$$

To show that this map is independent of the choice of $t \in T$, fix a section e of $\mathcal{E}_T$, and for each $t' \in T$, let $e_{t'}$ be its germ at t'. Let $T' := \{t' \in T : \mathrm{cosp}_{t',S}(e_{t'}) = \mathrm{cosp}_{t,S}(e_t)\}$. If $t' \in T$, choose a neighborhood V of t' satisfying condition (2) and an element e_V of $\mathcal{E}(V)$ whose germ at t' is $e_{t'}$. Then for every $t'' \in V \cap T$, $\mathrm{cosp}_{t'',S}(e_{t''})$ is the restriction of e_V to $\mathcal{E}(V \cap S) = \mathcal{E}_S$, since $V \cap T$ is connected. If t' belongs to the closure of T', we can choose a point $t'' \in V \cap T'$. Then for every point t''' of $V \cap T$, we have

$$\mathrm{cosp}_{t''',S}(e_{t'''}) = \mathrm{cosp}_{t'',S}(e_{t''}) = \mathrm{cosp}_{t,S}(e).$$

Thus $V \cap T \subseteq T'$, and it follows that T' is open, closed, and nonempty. Since T is connected, in fact $T' = T$.

Thus, for each pair T, S of elements of Π with $T \subseteq \overline{S}$, and for any $t \in T$ and $s \in S$, we have a cospecialization maps fitting into a commutative diagam:

$$\begin{tikzcd}
\mathcal{E}_T \arrow[r, "\mathrm{cosp}_{T,S}"] \arrow[d, "\cong"'] & \mathcal{E}_S \arrow[d, "\cong"] \\
\mathcal{E}_t \arrow[ur, "\mathrm{cosp}_{t,S}"] \arrow[r, "\mathrm{cosp}_{t,s}"'] & \mathcal{E}_s.
\end{tikzcd}$$

These cospecialization maps satisfy the following cocycle conditions: $\mathrm{cosp}_{S,S} = \mathrm{id}$ and, if $T \subseteq S^- \subseteq R^-$, then $\mathrm{cosp}_{T,R} = \mathrm{cosp}_{T,S} \circ \mathrm{cosp}_{S,R}$.

We shall say that a point x of X is a *central point* for Π if x belongs to the closure of every element of Σ. Any point x of X has a neighborhood U such that x is a central point for $\Sigma_{|U}$: it suffices to take a neighborhood U of x that meets only finitely many strata and then remove the closures of all the strata whose closures do not contain x.

Proposition 2.5.9. *Let $\mathcal{E}$ be a sheaf on a topological space X, let Π be a trivializing partition for $\mathcal{E}$ satisfying conditions (1)–(2) and (4) of Remark 2.5.3, and let x be a point of X.*

1. *If x is a central point for Π, then the natural map $\mathcal{E}(X) \to \mathcal{E}_x$ is an isomorphism.*

2. *Every point x has a neighborhood basis of open sets U such that each map $\mathcal{E}(U) \to \mathcal{E}_x$ is an isomorphism.*

Proof. Assume that x is a central point for Π. First we prove that the natural map $\mathcal{E}(X) \to \mathcal{E}_x$ is injective. The central point x belongs to the closure of the stratum containing every point y, so the map $\mathcal{E}(X) \to \mathcal{E}_y$ factors through the cospecialization map $\mathrm{cosp}_{x,y}$. Thus if two elements e, e' of $\mathcal{E}(X)$ have the same image in $\mathcal{E}_x$, they have the same image in $\mathcal{E}_y$ for every y, and hence must be equal. For the surjectivity, let e_x be an element of $\mathcal{E}_x$. The closure of the stratum containing every point y of X contains x, so the cospecialization map $\mathrm{cosp}_{x,y}$ is defined, and we let $e_y := \mathrm{cosp}_{x,y}(e_x)$; note that the definition of e_x is unambiguous because $\mathrm{cosp}_{x,x} = \mathrm{id}$. We claim that there is an element e of $\mathcal{E}(X)$ whose germ at every y is e_y. This claim can be verified locally on X. Given $y \in Y$, we can find an open neighborhood V of y and an element e' of $\mathcal{E}(V)$ such that $e'_y = e_y$. Shrinking V, we may assume that y is a central point of V and that the restriction of Π to V satisfies conditions (1)–(4). We claim then that $e'_z = e_z$ for every $z \in V$. The closure of the stratum containing z also contains x, so $\mathrm{cosp}_{x,z}$ is defined, and the cocycle condition tells us that

$$e_z := \mathrm{cosp}_{x,z}(e_x) = \mathrm{cosp}_{y,z}(\mathrm{cosp}_{x,y}(e_x) = \mathrm{cosp}_{y,z}(e_y) = \mathrm{cosp}_{y,z}(e'_y) = e'_z.$$

This completes the proof.

To prove statement (2), observe that if $x \in X$, then there is an open neighborhood U of x in which x is a central point. This remains true in every smaller open neighborhood, and furthermore the restriction of Π to every open neighborhood still satisfies the frontier condition (4). If U is chosen so that condition (2) also holds, then statement (1) implies that the map $\mathcal{E}(U) \to \mathcal{E}_x$ is an isomorphism.

□

Theorem 2.5.10. *An integral sheaf of monoids $\mathcal{M}$ on a topological space X is fine if it satisfies the following three conditions.*

1. *X admits an open covering on which $\overline{\mathcal{M}}$ admits a trivializing partition Π satisfying conditions (1)–(2) and (4) of Remark 2.5.3.*

2. *For each $x \in X$, the stalk $\overline{\mathcal{M}}_x$ of $\overline{\mathcal{M}}$ is finitely generated.*

3. *Whenever S and T are elements of Π with $T \subseteq \overline{S}$, the cospecialization map* $\operatorname{cosp}_{T,S} \colon \overline{\mathcal{M}}_T \to \overline{\mathcal{M}}_S$ *identifies $\overline{\mathcal{M}}_S$ with the quotient of $\overline{\mathcal{M}}_T$ by a face.*

Conversely, every fine sheaf of monoids $\mathcal{M}$ on a noetherian topological space satisfies these conditions.

Proof. Suppose that $\mathcal{M}$ satisfies the conditions (1) through (3) and let x be a point of X. Without loss of generality we may assume that x is a central point for the trivializing partition Σ. Since $\overline{\mathcal{M}}_x$ is finitely generated, $\mathcal{M}_x$ admits a markup $L \to \mathcal{M}_x^{\mathrm{gp}}$ and hence an exact chart $\beta_x \colon Q \to \mathcal{M}_x$, as explained in in Proposition 2.3.4. Since Q is finitely generated, Lemma 2.2.4 tells us that, after replacing X by an open neighborhood of x, we can find a homomorphism $\beta \colon Q \to \mathcal{M}$ whose stalk at x is β_x. Then β is a chart for $\mathcal{M}$ at x, which, by Proposition 2.1.4 means that the induced map $Q/F_x \to \overline{\mathcal{M}}_x$ is an isomorphism, where $F_x := \beta_x^{-1}(\mathcal{M}_x^*)$. We claim that the same is true for every point y of X. For any such point we have a commutative diagram

$$\begin{array}{ccc} Q/F_x & \xrightarrow{\overline{\beta}_x} & \overline{\mathcal{M}}_x \\ {\scriptstyle \pi_{x,y}}\downarrow & & \downarrow{\scriptstyle \operatorname{cosp}_{x,y}} \\ Q/F_y & \xrightarrow{\overline{\beta}_y} & \overline{\mathcal{M}}_y. \end{array}$$

By condition (3), $\operatorname{cosp}_{x,y}$ is the quotient of $\overline{\mathcal{M}}_x$ by a face G, which can only be $\operatorname{cosp}_{x,y}^{-1}(0)$. Since F_y is by definition the inverse image of 0 in Q, it follows that $\pi_{x,y}$ is the quotient of Q/F_x by $\overline{\beta}_x^{-1}(G)$, and hence that $\overline{\beta}_y$ is also an isomorphism.

Conversely, suppose that $\mathcal{M}$ is fine and that X is noetherian. Shrinking X, we may apply Corollary 2.3.6 to find a fine chart $P \to \mathcal{M}$. Let $h \colon X \to S := \operatorname{spec}(Q)$ be the corresponding map of locally monoidal spaces. Then by Proposition 2.1.4, $\overline{\mathcal{M}} \cong h^{-1}(\overline{\mathcal{M}}_S)$. Since S is a finite Kolmogoroff space, $\overline{\mathcal{M}}_S$ is quasi-constructible and, by Remark 2.5.2, $\overline{\mathcal{M}}$ admits a trivializing partition. Proposition 2.5.5 allow us to refine this into one satisfying properties (1)–(2), and (4) of Remark 2.5.3. Furthermore, properties (2) and (3) hold for $\overline{\mathcal{M}}_S$, and hence also for $\overline{\mathcal{M}}$. □

The Theorem 2.5.10 does not apply directly directly when X is a complex analytic space, since it may not be so easy to find a trivializing partition satisfying (1)–(2) and (4). One could try to work with partitions by complex analytic sets, in which case one can adapt the argument of Proposition 2.5.5 to show the existence of refinements satisfying the frontier condition. However one wants to work locally in the strong topology, and it becomes difficult to arrange for condition (2). For example, let X be the subset of $\mathbf{C}^3$ defined by the vanishing of $zy^2 - zx^2 + x^3$ (see Figure 1.6.2) on page 292 and let $\mathcal{M} \to \mathcal{O}_X$ be the log structure on X associated to the prelog structure $\mathbf{N} \to x - y$. Then $\overline{\mathcal{M}}$ is $\mathbf{N}$ on the z-axis T is 0 on $S := X \setminus T$. This stratification satisfies the frontier condition but not condition (2), and in fact it has no analytic refinement satisfying (2) in any neighborhood of the origin.

Proposition 2.5.11. *If $\mathcal{M}$ is a fine sheaf of monoids on a noetherian and sober space X and U is an open subset of X, then $\Gamma(U, \overline{\mathcal{M}}_X)$ is fine.*

Proof. It suffices to treat the case $U = X$. By Theorem 2.5.10 and Proposition 2.5.9, every point x admits an open neighborhood U_x such that the map $\overline{\mathcal{M}}_X(U_x) \to \overline{\mathcal{M}}_{X,x}$ is an isomorphism. In particular, $\overline{\mathcal{M}}_X(U_x)$ is a fine monoid. Since X is quasi-compact, there exists a finite set $\{U_{x_1}, \ldots, U_{x_n}\}$ of these neighborhoods that covers X. We prove that $\Gamma(U_m, \overline{\mathcal{M}}_X)$ is fine by induction on m, where $U_m := \cup\{U_{x_i} : i \leq m\}$. In fact, $\Gamma(U_m, \overline{\mathcal{M}}_X)$ is the fiber product of $\Gamma(U_{m-1}, \overline{\mathcal{M}}_X)$ and $\Gamma(U_{x_m}, \overline{\mathcal{M}}_X)$ over the integral monoid $\Gamma(U_{m-1} \cap U_{x_m}, \overline{\mathcal{M}}_X)$, so it is fine by statement (6) of Theorem I.2.17. $\square$

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