Exercises on Log Geometry

Arthur Ogus

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1. Let $\theta_1 \colon P \to Q_1$ denote the monoid homomorphism $\mathbf{N} \oplus \mathbf{N} \to \mathbf{N} \oplus \mathbf{N}$ sending (m,n) to $(m+n,n)$ and let $\theta_2 \colon P \to Q_2$ denote the monoid homomorphism $\mathbf{N} \oplus \mathbf{N} \to \mathbf{N} \oplus \mathbf{N}$ sending (m,n) to $(m, m+n)$. Let $Q_1 \oplus_P Q_2$ be the pushout in the category of monoids and let $Q' := (Q_1 \oplus_P Q_2)^{int}$. Show that θ_1 and θ_2 are local but not exact. Show that the pushout $Q_1 \to Q_1 \oplus_Q Q_2$ is local but that the integral pushout $Q_1 \to Q'$ is not local.

2. Continuing with the notation of the previous problem, note that the monoid P has faces $F_0 := \{(0,0)\}$, $F_1 := \{(0,n)\}$, $F_2 := \{(m,0)\}$, and $F := P$, we use the same notation for the faces of the Q_i's, with G in place of F.

   (a) Find an isomorphism $\phi \colon Q_1^{\mathrm{gp}} \to Q_2^{\mathrm{gp}}$ such that $\phi \circ \theta_1 = \phi_2$. Find minimal faces of each Q_i such that ϕ induces an isomorphism on the localizations of each Q_i by the corresponding face.

   (b) Compute the mapping $\mathrm{Spec}(Q_i) \to \mathrm{Spec}(P)$ induced by θ_i.

   (c) Compute the mapping $H(\theta_i) \colon HQ_i) \to H(P)$ induced by θ_i. Draw a picture of this mapping that is consistent with your answer to (b). Show where $H(Q_i/G_j)$ goes.

3. Let $\theta \colon P \to Q$ be a homomorphism of fine monoids. Assume that the ideal K_θ of Q generated by the image of P^+ is reduced and that θ is is locally exact. Prove that θ is integral. (Note: It is not difficult to reduce to the case in which θ is local and P is sharp. You may restrict attention to this case if you like.)