Exercises on Log Geometry

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1. Let A be a noetherian local domain with maximal ideal $\mathfrak{m}$ and infinite residue field k. Assume that the dimension of $V := \mathfrak{m}/\mathfrak{m}^2$ is at least two. Prove that the monoid $\overline{A}' := (A \setminus \{0\})/A^*$ of effective divisions of A is not finitely generated. In particular, conclude that if A has dimension one and is not regular, then $\overline{A}'$ is not finitely generated.
   Hint: Consider the mapping $\overline{A}' \to (A/\mathfrak{m}^2)/A^*$ and the irreducible elements of the monoid $\overline{A}'$.

2. Let P be a (commutative) monoid and let S be a P-set. Suppose that S satisfied the following condition: * If $p \in P$ and $s \in S$ with $p + s = s$, then p is a unit. Prove that if S is finitely generated as a P-set and satisfies *, then it is generated by $S \setminus (P^+S)$. (Nakayama's lemma for P-sets.). Give examples showing that the hypotheses * and finite generation are not superfluous. Extra bonus question: Find a good terminology for the condition *.

3. Let $Q_{2,2}$ be the monoid with generators q_1, q_2, q_3, q_4 and relation $q_1 + q_2 = q_3 + q_4$. Then $H(Q_{2,2}) := \mathrm{Hom}(Q_{2,2}, \mathbf{N})$ is in a natural way embedded in $\mathbf{N}^4$. Find the minimal set of generators of $H(Q_{2,2}) \subseteq \mathbf{N}^4$.