This note explains a property of A_{inf} used in the paper "Revisiting the de Rham–Witt complex" by Bhatt–Lurie–Mathew.

Let C denote a perfectoid field of mixed characteristic $(0,p)$ that contains all the pth power roots of unity. We denote the valuation ring of C by $\mathcal{O}_C$, the maximal ideal of $\mathcal{O}_C$ by $\mathfrak{m}$, the residue field of $\mathcal{O}_C$ by k, and the ring of p-typical Witt vectors by $W = W(k)$.

Let $\mathcal{O}_C^\flat = \varprojlim_\varphi \mathcal{O}_C/p$ denote the tilt of $\mathcal{O}_C$. There is a well-defined multiplicative map

$$\sharp \colon \mathcal{O}_C^\flat \to \mathcal{O}_C$$

defined as follows: let $x = (x_0, x_1, \ldots) \in \mathcal{O}_C^\flat$. Choose a lift $\tilde{x}_n \in \mathcal{O}_C$ of $x_n \in \mathcal{O}_C/p$ and set

$$x^\sharp = \lim_{n\to\infty} \tilde{x}_n^{p^n}.$$

This limit exists and does not depend on the choice of the lifts. Then $\mathcal{O}_C^\flat$ is a valuation ring with respect to the norm $|\cdot|^\flat$ defined by

$$\mathcal{O}_C^\flat \ni x \mapsto |x|^\flat := |x^\sharp|_p,$$

where $|\cdot|_p$ is the p-adic norm on $\mathcal{O}_C$ normalized by $|p|_p = 1/p$. Note that the residue field of $\mathcal{O}_C^\flat$ is k.

Let A_{inf} denote the ring of p-typical Witt vectors $W(\mathcal{O}_C^\flat)$. The Frobenius of $\mathcal{O}_C^\flat$ induces an automorphism of A_{inf}, which we denote by φ.

Fix a system of primitive p^nth roots of unity $\epsilon_{p^n} \in \mu_{p^n}(\mathcal{O}_C)$ satisfying $\epsilon_{p^n}^p = \epsilon_{p^{n-1}}$. Let $\underline{\epsilon}$ denote the element

$$(1 \bmod p, \epsilon_p \bmod p, \epsilon_{p^2} \bmod p, \ldots) \in \mathcal{O}_C^\flat$$

and set

$$\mu = [\underline{\epsilon}] - 1 \in A_{\mathrm{inf}}.$$

Observe that $\varphi^{-1}(\mu) = [\underline{\epsilon}^{1/p}] - 1$ divides μ. More generally, $\varphi^{-r}(\mu)$ divides $\varphi^{-(r-1)}(\mu)$, and it induces a homomorphism $A_{\mathrm{inf}}/\varphi^{-(r-1)}(\mu) \to A_{\mathrm{inf}}/\varphi^{-r}(\mu)$.

The goal of this note is to prove the following:

Proposition 0.1. *The p-adic completion of $\varinjlim_r A_{\mathrm{inf}}/\varphi^{-r}(\mu)$ is isomorphic to W.*

Remark 0.2. As the proof will show, the isomorphism depends on the choice of a homomorphism $\mathcal{O}_C^\flat \to k$. Since $\mathcal{O}_C^\flat$ is a valuation ring with residue field k, we have the quotient map $\mathcal{O}_C^\flat \to k$. Note, however, on may sometimes use the composite of this quotient map and φ^{-1}. In what follows, we use the quotient homomorphism $\mathcal{O}_C^\flat \to k$ for simplicity.

Proof. Note first that taking the inductive limit of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \big(\varphi^{-(r-1)}(\mu)\big) & \longrightarrow & A_{\mathrm{inf}} & \longrightarrow & A_{\mathrm{inf}}/\big(\varphi^{-(r-1)}(\mu)\big) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \big(\varphi^{-r}(\mu)\big) & \longrightarrow & A_{\mathrm{inf}} & \longrightarrow & A_{\mathrm{inf}}/\big(\varphi^{-r}(\mu)\big) & \longrightarrow & 0
\end{array}$$

yields the isomorphism

$$\varinjlim_r A_{\mathrm{inf}}/\varphi^{-r}(\mu) \cong A_{\mathrm{inf}}/\bigcup_r \big(\varphi^{-r}(\mu)\big).$$

Let $\big(A_{\mathrm{inf}}/\bigcup_r \big(\varphi^{-r}(\mu)\big)\big)^\wedge$ denote its p-adic completion.

Consider the ring homomorphism $A_{\mathrm{inf}} \to W$ induced by the quotient homomorphism $\mathcal{O}_C^\flat \to k$. Since

$$|\underline{\epsilon} - 1|^\flat = \lim_{n\to\infty} |\epsilon_{p^n} - 1|_p^{p^n} = \frac{1}{p^{p/(p-1)}} < 1,$$

we have $\underline{\epsilon} - 1 \in \mathrm{Ker}(\mathcal{O}_C^\flat \to k)$. It follows $\underline{\epsilon}^{1/p^r} - 1 = \varphi^{-r}(\underline{\epsilon} - 1) \in \mathrm{Ker}(\mathcal{O}_C^\flat \to k)$ for every r. Hence $\varphi^{-r}(\mu) = [\underline{\epsilon}^{1/p^r}] - 1 \in \mathrm{Ker}(A_{\mathrm{inf}} \to W)$ by functoriality of the Teichmüller lifts. In particular, the homomorphism $A_{\mathrm{inf}} \to W$ induces

$$A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr) \to W.$$

Since W is p-adically complete, the above map induces

$$f\colon \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge \to W.$$

We prove that f is an isomorphism. First we prove that f is an isomorphism after reduction modulo p. On the one hand, observe that $f \bmod p$ is

$$\mathcal{O}_C^\flat/\bigcup_r \bigl(\underline{\epsilon}^{1/p^r} - 1\bigr) \to k.$$

On the other hand, $\bigcup_r \bigl(\underline{\epsilon}^{1/p^r} - 1\bigr) = \mathrm{Ker}(\mathcal{O}_C^\flat \to k)$: to see this, observe

$$|\underline{\epsilon}^{1/p^r} - 1|^\flat = \frac{1}{p^{1/(p^{r-1}(p-1))}} \to 0 \quad (r \to \infty).$$

Hence $f \bmod p$ is an isomorphism.

Since the source and the target of f are both p-adically complete and since $f \bmod p$ is surjective, it follows that f is surjective: take any $x \in W$. Since $f \bmod p$ is surjective, there exist $a_0 \in \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge$ and $x_0 \in W$ such that

$$x - f(a_0) = px_0.$$

Repeating this arguments gives $a_n \in \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge$ and $x_n \in W$ such that

$$x_{n-1} - f(a_n) = px_n.$$

Then

$$a = \sum_{n=0}^{\infty} p^n a_n \in \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge$$

satisfies $f(a) = x$.

Since W is p-torsion free, $\bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge$ is p-adically separated, and f mod p is injective, it follows that f is injective: take any $a \in \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge$ with $f(a) = 0$. Since $f \bmod p$ is injective, there exists $a_1 \in \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge$ with $a = pa_1$. Then $pf(a_1) = f(a) = 0$. Since W is p-torsion free, we have $f(a_1) = 0$. Thus there exists $a_2 \in \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge$ with $a_1 = pa_2$. Repeating this argument gives

$$a \in \bigcap_n p^n \bigl(A_{\mathrm{inf}}/\bigcup_r \bigl(\varphi^{-r}(\mu)\bigr)\bigr)^\wedge = \{0\}.$$

From these arguments we conclude that f is an isomorphism. $\square$