The Nygaard filtration of a strict Dieudonné complex

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Our aim here is to give a brief account of some of the essential features of the construction of the Nygaard filtration as discussed in [1]. We also explain its application to the proof of Katz's conjecture, following Nygaard's method in [4], but adapted to the language of [1].

We begin with a general construction, going back to Mazur's original article [3]. Let p be a fixed prime number.

Definition 1 *Let $\Phi\colon M' \to M$ be an injective homomorphism of p-torsion free complexes of abelian sheaves on a topological space X. Let $\overline{M} := M/pM$, and define, for $i \geq 0$,*

$$\begin{array}{rcl} N^iM' & := & \Phi^{-1}(p^iM) \\ N_iM & := & Im(p^{-i}\Phi\colon N^iM' \to M) \\ N^i\overline{M}' & := & Im(N^iM' \to M'/pM') \\ N_i\overline{M} & := & Im(N_iM \to M/pM) \end{array}$$

The verification of the following proposition is immediate.

Proposition 2 *With the definitions above, $N^{\cdot}$ is a descending filtration of M' and $N_{\cdot}$ is an ascending filtration of M. Furthermore*

$$\begin{array}{rcl} pN^{i-1}M' & = & N^iM' \cap pM' \\ pN_{i+1}M & = & N_iM \cap pM \end{array}$$

The map $p^{-i}\Phi$ induces isomorphisms of pairs

$$(N^iM', N^{i+1}M') \longrightarrow (N_i, pN_{i+1})$$

$$(N^iM', pN^{i-1}M') \longrightarrow (N_i, N_{i-1}),$$

and hence isomorphisms:

$$\mathrm{Gr}^i_N M' \longrightarrow N_i\overline{M}$$

$$N^i\overline{M}' \longrightarrow \mathrm{Gr}^N_i M$$

$$N^iM'/(N^{i+1}M' + pN^{i-1}M') \longrightarrow \mathrm{Gr}^i_N \overline{M}'$$

$$N_iM/(N_{i-1}M + pN_{i_1}M) \longrightarrow \mathrm{Gr}^N_i \overline{M}$$

□

Example 3 Let W be the Witt ring of a perfect field k. Following Mazur, let us define a "span" to be an injective homomorphism $\Phi\colon M' \to M$ of finitely generated W-modules of the same rank. For example, let i be a natural number and let $\Phi\colon W \to W$ denote multiplication by p^i. Then $N^{\cdot}\overline{M}'$ (resp. $N.\overline{M}$) is the unique filtration on k such that $\mathrm{Gr}^i k$ (resp. $\mathrm{Gr}_i k$) is nonzero. It is standard fact that every span is in fact a direct sum of spans of this form. Thus a span is determined up to isomorphism by the "abstract Hodge numbers" $h^i(\Phi) := \dim_k \mathrm{Gr}^i_N \overline{M}' = \dim_k \mathrm{Gr}^N_i \overline{M}$.

Now suppose that $(M^{\cdot}, d, F)$ is a saturated Dieudonné complex and let

$$\Phi\colon (M^{\cdot}, d) \to (M^{\cdot}, d)$$

be the corresponding morphism of complexes. We assume here that $M^n = 0$ for $n < 0$, so $\Phi^n = p^nF$. Then it is easy to describe the filtrations $N^{\cdot}$ and $N.$ explicitly.

Proposition 4 *Let $(M^{\cdot}, d, F)$ be a saturated Dieudonné complex and let $N^{\cdot}$ and $N.$ be the filtrations on $M^{\cdot}$ defined by Φ as in Definition 1. Then*

$$N^iM = p^{i-1}VM^0 \to p^{i-2}VM^1 \to \cdots \to VM^{i-1} \to M^i \to M^{i+1} \cdots$$

$$N_iM \;=\; M^0 \;\to\; M^1 \to \;\cdots \to\; M^{i-1} \;\to\; FM^i \to pFM^{i+1} \to \cdots$$

Furthermore, the inverse of the isomorphism $p^{-i}\Phi\colon N^iM' \to N_iM$ is given by $p^{i-n-1}V$ in degree n.

Proof: An element x of M^n lies in N^iM^n if and only if $p^nFx = p^iy$ for some $y \in M^n$ Thus $N^iM^n = M^n$ when $i \leq n$, and when $n < i$, if and only if $Fx = p^{i-n-1}py = p^{i-n-1}FVy$, that is, if and only if $x = p^{i-n-1}Vy$ for some y. Furthermore, $p^{-i}\Phi p^{i-n-1}Vy = p^{n-i}Fp^{i-n-1}Vy = y$ for every $y \in M^n$, so $N_iM^n = M^n$ when $n < i$, and if $i \leq n$, then $p^{-i}\Phi N^iM^n = p^{n-i}FM^n$. □

The following result corresponds to Nygaard's [4, Theorem 1.5]. The first statement occurs in [1, Proposition 8.2.1], but not the second. (Actually Nygaards' theorem is more general, and applies to powers of Φ as well as to Φ.)

Theorem 5 *There are natural quasi-isomorphisms:*

$$
\begin{array}{ccccccccc}
N^i\overline{M}^{\cdot} & = & \cdots 0 & \longrightarrow & VM^{i-1}/pM^{i-1} & \longrightarrow & \overline{M}^i & \longrightarrow & \overline{M}^{i+1}\cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\beta^{\geq i}\mathcal{W}_1 M^{\cdot} & = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{W}_1 M^i & \longrightarrow & \mathcal{W}_1 M^{i+1}
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
N_i\overline{M}^{\cdot} & = & \overline{M}^0 & \longrightarrow \cdots & \overline{M}^{i-1} & \longrightarrow & FM^i/pM^i & \longrightarrow & 0\cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\tau^{\leq i}\overline{\mathcal{W}}_1 M^{\cdot} & = & \mathcal{W}_1 M^0 & \longrightarrow \cdots & \mathcal{W}_1 M^{i-1} & \longrightarrow & Z^i(\mathcal{W}^1 M^{\cdot}) & \longrightarrow & 0\cdots
\end{array}
$$

Proof: Since $N^i\overline{M} = N^iM/(N^iM \cap pM) = N^iM/pN^{i-1}M$, the description of $N^i\overline{M}^{\cdot}$ shown follows from Proposition 4, and similarly for the description of $N_i\overline{M}$. Now recall from [1, Corollary 2.7.2] that the natural surjection $\pi\colon \overline{M}^{\cdot} \to \mathcal{W}_1 M^{\cdot}$ is a quasi-isomorphism, *i.e.*, its kernel $K^{\cdot}$ is acyclic. We claim that the same is true for the surjection $\pi'\colon N^i\overline{M} \to \beta^{\geq i}\mathcal{W}_1 M^{\cdot}$, with kernel K'. First we check degree $i-1$, where we need to show that the map $V\overline{M}^{i-1} \to \overline{M}^i$ is injective. Suppose that $x \in M^{i-1}$ and $dVx = py$ with $y \in M^i$. Then $dx = FdVx = Fpy = pFy$, and since $M^{\cdot}$ is saturated, it follows that $x = Fx'$ for some $x' \in M^{i-1}$. But then $Vx = px'$ so Vx maps to zero in $\overline{M}^{i-1}$. Now let us check that the map is an isomorphism in degrees $j \geq i$. From the exact sequence $0 \to K'^{\cdot} \to N^i\overline{M}^{\cdot} \to \mathcal{W}_1 M^{\cdot} \to 0$ we see that it is enough to check that $H^j(K'^{\cdot})$ for $j \geq i$. Since $K^{\cdot}$ is acyclic, $H^n(K'^{\cdot}) \cong H^{n-1}(K^{\cdot}/K'^{\cdot})$ for all n, so we just need to show that $H^j(K^{\cdot}/K'^{\cdot}) = 0$ for $j \geq i-1$. The complex $K^{\cdot}/K'^{\cdot}$ vanishes in degrees $\geq i$, so it suffices to check degree $i-1$. Recall that $K^{i-1} = V\overline{M}^{i-1} + dV\overline{M}^{i-2}$ and $K'^{i-1} = V\overline{M}^{i-1}$. Thus the boundary map $K^{i-2}/K'^{i-2} \to K^{i-1}/K'^{i-1}$ is surjective and hence there is no cohomology in degree $i-1$. This completes the proof that the map $N^i\overline{M}^{\cdot} \to \beta^{\geq i}\mathcal{W}_1 M^{\cdot}$ is a quasi-isomorphism

For the second diagram, recall that if $x \in M^i$ and $dx \in pM^{i+1}$, then $x \in FM^i$. Thus $F\overline{M}^i$ identifies with $Z^i(\overline{M}^\cdot)$ and $N.\overline{M}^\cdot$ with $\tau^{\leq i}\overline{M}^\cdot$. Since $\overline{M}^\cdot \to \overline{\mathcal{W}}_1 M^\cdot$ is a quasi-isomorphism, the same holds after applying $\tau^{\leq i}$ and the result follows. □

The following result shows that, under suitable hypotheses, formation of the filtrations $N^\cdot$ and $N.$ commutes with passage to hypercohomology.

Theorem 6 *Let $(M^\cdot, F, d)$ be a strict Dieudonné complex on a topological space (or topos) X, let $H^\cdot := H^\cdot(M^\cdot, d)$ and suppose that the following hypotheses are satisfied.*

1. *The groups in $H^\cdot$ are p-torsion free.*
2. *The two spectral sequences of hypercohomology associated to the complex $\mathcal{W}_1 M^\cdot$ degenerate, at E_1 and at E_2 respectively. That is:*
   (a) *For all i, the map $H^\cdot(X, \beta^{\geq i}\mathcal{W}_1 M^\cdot) \to H^\cdot(X, \mathcal{W}_1 M^\cdot)$ are injective.*
   (b) *For all i, the maps $H^\cdot(X, \tau^{\leq i}\mathcal{W}_1 M^\cdot) \to H^\cdot(X, \mathcal{W}_1 M^\cdot)$ are injective.*

Let $N^i H^\cdot$ and $N_i H^\cdot$ be the submodules of $H^\cdot$ defined by the map $H^\cdot(\Phi)\colon H^\cdot \to H^\cdot$ as in Definition 1. Then the following conclusions hold.

1. *For all i, the natural maps*
   $$H^\cdot(M^\cdot)/p^i H^\cdot(M^\cdot) \to H^\cdot(M^\cdot/p^i M^\cdot)$$
   are isomorphisms. In particular, the natural maps
   $$\overline{H}^\cdot := H^\cdot/pH^\cdot \to H^\cdot(\overline{M}^\cdot) \to H(\overline{\mathcal{W}}_1 M^\cdot)$$
   are isomorphisms.
2. *The natural maps*
   $$H^\cdot(N^i M^\cdot) \to N^i H^\cdot(M^\cdot) \quad \text{and} \quad H^\cdot(N_i M^\cdot) \to N_i H^\cdot(M^\cdot)$$
   are isomorphisms.
3. *The natural maps*
   $$H^\cdot(N^i M^\cdot) \to H^\cdot(\beta^{\geq i}\mathcal{W}_1 M^\cdot) \quad \text{and} \quad H^\cdot(N_i M^\cdot) \to H^\cdot(\tau^{\leq i}\mathcal{W}_1 M^\cdot)$$
   are surjective.

Proof: Conclusion (1) follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to M^{\cdot} \xrightarrow{p^i} M^{\cdot} \to M^{\cdot}/p^i M^{\cdot} \to 0,$$

hypothesis (1), and the fact that $\overline{M}^{\cdot} \to \mathcal{W}_1 M^{\cdot}$ is a quasi-isomorphism.

Lemma 7 *For every i, the map $H^{\cdot}(N^i M^{\cdot}) \to H^{\cdot}(M^{\cdot})$ is injective.*

Proof: We use induction on i, the case $i = 0$ being trivial. Thanks to Proposition 2, we have an exact sequence

$$0 \to N^{i-1} M^{\cdot} \xrightarrow{[p]} N^i M^{\cdot} \to N^i \overline{M}^{\cdot} \to 0 \tag{1}$$

and hence a commutative diagram in which the rows are exact:

$$\begin{array}{ccccc} H^{\cdot}(N^{i-1}M^{\cdot}) & \xrightarrow{[p]} & H^{\cdot}(N^i M^{\cdot}) & \longrightarrow & H^{\cdot}(N^i \overline{M}^{\cdot}) \\ \downarrow a_{i-1} & & \downarrow a_i & & \downarrow b_i \\ H^{\cdot}(M^{\cdot}) & \xrightarrow{p} & H^{\cdot}(M^{\cdot}) & \longrightarrow & H^{\cdot}(\overline{M}^{\cdot}). \end{array}$$

The map a_{i-1} is injective by the induction hypothesis, the map p in the lower left is injective because $H^{\cdot}(M)$ is torsion free, and by Theorem 5 the map b_i identifies with the map $H^{\cdot}(\beta^{\geq i}\mathcal{W}_1 M^{\cdot}) \to H^{\cdot}(\mathcal{W}_1 M^{\cdot})$ which is injective by hypothesis (2a). It follows that a_i is injective. □

Since $N^i M^{\cdot}$ is the kernel of the map

$$M^{\cdot} \xrightarrow{\Phi} M^{\cdot} \to M^{\cdot}/p^i M^{\cdot},$$

we find a map

$$\phi_i \colon M^{\cdot}/N^i M^{\cdot} \to M^{\cdot}/p^i M^{\cdot}$$

Lemma 8 *For every i, the map $H^{\cdot}(M^{\cdot}/N^i M^{\cdot}) \to H^{\cdot}(M^{\cdot}/p^i M^{\cdot})$ induced by ϕ_i is injective.*

Proof: We argue by induction on i, the case $i = 0$ being trivial. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gr}^i_N M^{\cdot} & \longrightarrow & M^{\cdot}/N^{i+1}M^{\cdot} & \longrightarrow & M^{\cdot}/N^i & \longrightarrow & 0 \\ & & \downarrow \psi_i & & \downarrow \phi_{i+1} & & \downarrow \phi_i & & \\ 0 & \longrightarrow & M^{\cdot}/pM^{\cdot} & \xrightarrow{[p^i]} & M^{\cdot}/p^{i+1}M^{\cdot} & \longrightarrow & M^{\cdot}/p^i M^{\cdot} & \longrightarrow & 0 \end{array}$$

with exact rows. Furthermore, the map ψ_i factors as a composition

$$\operatorname{Gr}^i_N M^{\cdot} \xrightarrow{\alpha_i} N_i\overline{M}^{\cdot} \to \overline{M}^{\cdot}$$

where the first arrow is the isomorphism from Proposition 2 and the second is the evident inclusion. This yields the diagram:

$$\begin{array}{ccccc}
H^{\cdot}(\operatorname{Gr}^i_N M^{\cdot}) & \longrightarrow & H^{\cdot}(M^{\cdot}/N^{i+1}M^{\cdot}) & \longrightarrow & H^{\cdot}(M^{\cdot}/N^iM^{\cdot}) \\
\Big\downarrow \psi_i & & \Big\downarrow \phi_{i+1} & & \Big\downarrow \phi_i \\
H^{\cdot}(\overline{M}^{\cdot}) & \xrightarrow{[p^i]} & H^{\cdot}(M^{\cdot}/p^{i+1}M^{\cdot}) & \longrightarrow & H^{\cdot}(M^{\cdot}/p^iM^{\cdot}).
\end{array}$$

The rows in the diagram are exact, the map labeled $[p^i]$ is injective by hypothesis (1), and the map ϕ_i is injective by the induction hypothesis. The map ψ_i factors as a composite

$$H^{\cdot}(\operatorname{Gr}^i_N M^{\cdot}) \xrightarrow{H^{\cdot}(\alpha_i)} H^{\cdot}(N_i\overline{M}^{\cdot}) \xrightarrow{\beta_i} H^{\cdot}(\overline{M}^{\cdot});$$

The first map is an isomorphism since α_i is, and by Theorem 5, β_i identifies with the map $H^{\cdot}(X, \tau^{\le i}\mathcal{W}_1 M^{\cdot}) \to H^{\cdot}(X, \mathcal{W}_1 M^{\cdot})$, which is injective by hypothesis (2b). It follows that ψ_i is injective and then that ϕ_{i+1} is injective. □

Lemma 9 *The map $H^{\cdot}(N^iM^{\cdot}) \to H^{\cdot}(N^i\overline{M}^{\cdot})$ is surjective.*

Proof: The exact sequence (1) yields a long exact sequence

$$H^{\cdot}(N^iM^{\cdot}) \to H^{\cdot}(N^i\overline{M}^{\cdot}) \to H^{\cdot+1}(N^{i-1}M^{\cdot}) \xrightarrow{[p]} H^{\cdot+1}(N^iM^{\cdot}).$$

Thus it suffices to show that the map $[p]$ is injective. This follows from the commutative diagram

$$\begin{array}{ccc}
H^{\cdot}(N^{i-1}M^{\cdot}) & \xrightarrow{[p]} & H^{\cdot}(N^iM^{\cdot}) \\
\Big\downarrow & & \Big\downarrow \\
H^{\cdot}(M^{\cdot}) & \xrightarrow{p} & H^{\cdot}(M^{\cdot}),
\end{array}$$

the torsion freeness of $H^{\cdot}(M^{\cdot})$, and Lemma 7. □

Now to prove the theorem, recall that $N^iH^\cdot$ is by definition the kernel of the composition

$$c_i \colon H^\cdot(M^\cdot) \xrightarrow{H^\cdot(\Phi)} H^\cdot(M^\cdot) \longrightarrow H^\cdot(M^\cdot)/p^iH^\cdot(M^\cdot).$$

The top row of the following commutative diagram is exact:

$$\begin{array}{ccccc}
H^\cdot(N^iM^\cdot) & \xrightarrow{a_i} & H^\cdot(M^\cdot) & \longrightarrow & H^\cdot(M^\cdot/N^iM^\cdot) \\
 & & \Big\downarrow c_i & & \Big\downarrow \phi_i \\
 & & H^\cdot(M^\cdot)/p^iH^\cdot(M^\cdot) & \xrightarrow{\cong} & H^\cdot(M^\cdot/p^iM^\cdot).
\end{array}$$

As we have seen, a_i and ϕ_i are injective, and it follows that $H^\cdot(N^iM^\cdot)$ identifies with the kernel of c_i. □

Let us sketch how Theorem 6 implies Katz's conjecture. Recall that if X/k is a smooth over a perfect field k of characteristic p, the classical de Rham Witt complex $W\Omega^\cdot_X$ identifies with the strict de Rham Witt complex $\mathcal{W}\Omega^\cdot_X$ constructed in [1] and that its hypercohomology identifies with crystalline cohomology [2].

Theorem 10 *Let X/k be a smooth proper scheme over a perfect field k of characteristic $p > 0$ and let $H^\cdot_{dRW}(X) := H^\cdot(X, \mathcal{W}\Omega^\cdot_X)$. Assume that $H^\cdot_{dRW}(X/W)$ is torsion free and that the Hodge spectral sequence of of X/k degenerates at E_1. Let Φ denote the endomorphism of $H^\cdot_{dRW}(X)$ induced by F_X and let $N^\cdot$ and $N.$ be the corresponding filtrations of $H^\cdot_{dRW}(X)$ as in Definition 1. Then*

1. *The natural map $\overline{H}^\cdot := H^\cdot_{dRW}(X)/pH^\cdot_{dRW}(X) \to H^\cdot_{dR}(X/k)$ is an isomorphism.*

2. *The filtration induced by $N^\cdot$ on $H_{dR}(X/k)$ is the Hodge filtration.*

3. *The filtration induced by $N.$ on $H_{dR}(X/k)$ is the conjugate filtration.*

4. *The dimension of $\mathrm{Gr}^i_N \overline{H}^n$ is equal to the dimension of $H^{n-i}(X, \Omega^i_{X/k})$.*

5. *The Newton polygon of the action of Φ on $H^\cdot_{dRW}(X)$ lies on or above the Hodge polygon of X/k in degree n.*

Proof: Statements (1)–(4) follow from Theorem 6 appied to the saturated de Rham Witt complex $\mathcal{W}\Omega^{\cdot}_X$ and the isomorphism $\Omega^{\cdot}_{X/k} \cong \mathcal{W}_1\Omega^{\cdot}_X$ of [1, Proposition 4.3.2]. Statement (5) follow, since the Newton polygon of an F-crystal always lies on or above the polygon formed from the numbers $\dim \mathrm{Gr}^i_N \overline{H}$ [3]. □

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