The following result is used in the proof of Lemma 10.3.8 of [1].

Proposition 1 *Let D' denote the full subcategory of the derived category D of W-modules such that $H^i(C \otimes^L k) = 0$ for $i < 0$. Then on D' there is a natural transformation $L\eta \to \mathrm{id}$ extending the usual construction on the set of complexes with $C^i = 0$ for $i < 0$.*

Proof: Since W has finite homological dimension, every object C of D
is isomorphic to an object whose terms are free. Thus we may restrict our attention to such objects.

Lemma 2 *Let C be a complex of torsion-free W-modules. Then $H^i(C \otimes k)$ vanishes for all $i < 0$ if and only if multiplication by p is bijective on $H^i(C)$ for all $i < 0$ and injective on $H^0(C)$.*

Proof: The exact sequence $0 \to C \xrightarrow{p} C \longrightarrow C \otimes k \to 0$ yields an exact sequence:

$$\cdots \to H^{i-1}(C \otimes k) \longrightarrow H^i(C) \xrightarrow{p} H^i(C) \to H^i(C \otimes k) \longrightarrow \cdots$$

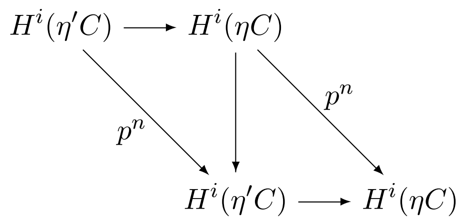
□

Let C be a complex satisfying the conditions of the lemma. By definition, ηC is the subcomplex of $\mathbf{Q} \otimes C$ which in degree i is $\{x \in p^i C^i : dx \in p^{i+1} C^{i+1}\}$. Let $\eta' C := C \cap \eta C$, *i.e.*, the subcomplex of ηC which is C^i in degree $i < 0$ and is ηC in degree $i \geq 0$. We have arrows: $\eta' C \to C$ and $\eta' C \to \eta C$. Recall from [1, 7.2.1] that for each i, there is a natural isomorphism $H^i(C)/[p] \to H^i(\eta C)$ for all i. Thus if C satisfies the conditions of the lemma, so does ηC. Note also that the map $H^i(\eta' C) \to H^i(C)$ is an isomorphism for $i < 0$, so it still true that multiplication by p is bijective on $H^i(\eta' C)$ for $i < 0$. Furthermore, $H^0(\eta' C) \to H^0(C)$ is an isomorphism, so $H^0(\eta' C)$ is also torsion free.

To obtain our morphism in the derived category, is enough to prove that if C satisfies the hypothesis of the lemma, then $\eta' C \to \eta C$ is a quasi-isomorphism. In degree 0 this is true because of the commutative diagram:

$$\begin{array}{ccc} H^0(\eta' C) & \xrightarrow{\cong} & H^0(C) \\ \downarrow & & \downarrow \\ H^0(\eta C) & \xleftarrow{\cong} & H^0(C)/[p] \end{array}$$

and the fact that $H^0(C)$ is torsion free. To prove it in negative degrees, we may argue degree by degree, and so we can assume that C is bounded below. Then for $n >> 0$, multiplication by p^n on ηC factors through $\eta' C$, and we get a diagram



Since the slanted maps are isomorphisms, so are the horizontal ones. □

References

[1] B. Bhatt, J. Lurie, and A. Matthew. Revisiting the de Rham Witt complex. arXiv:1804.05501v1.