The de Rham Witt complex and crystalline cohomology

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If $X/k$ is a smooth projective scheme over a perfect field $k$, let us try to find an explicit quasi-isomorphism $\text{Ru}_{X/W^*}(\mathcal{O}_{X/S}) \cong \mathcal{W}\Omega_X$. To do this we need an explicit representative of $\text{Ru}_{X/W^*}(\mathcal{O}_{X/S})$ together with its Frobenius action. The standard way to do this is to choose an embedding $X \rightarrow \tilde{Y}$, where $\tilde{Y}/W$ is smooth and endowed with a lift $\phi_{\tilde{Y}}$ of Frobenius. For example, if $X$ is quasi-projective, we can let $\tilde{Y}$ be a projective space $\mathbb{P}^N$ endowed with the endomorphism defined by raising coordinates to their $p$th power. (If $X$ is not projective, one can use local liftings and simplicial methods; which we shall not discuss here.) Once such an embedding $(\tilde{Y}, \phi_{\tilde{Y}})$ is chosen, let $Y$ be its reduction modulo $p$, let $\tilde{D}$ denote the ($p$-adically complete) PD-envelope of $X$ in $\tilde{Y}$ and let $D$ be its reduction modulo $p$. Then the $\mathcal{O}_{\tilde{Y}}$-module $\mathcal{O}_{\tilde{D}}$ admits an integrable connection [1, 6.4], whose corresponding de Rham complex $\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^{*}$ is a representative of $\text{Ru}_{X/W^*}(\mathcal{O}_{X/W})$ [1, 7.1]. The assumed lifting $\phi_{\tilde{Y}}$ of $F_Y$ extends uniquely to a PD-morphism $\phi_D$ of $\tilde{D}$. This morphism induces an endomorphism $\phi_{\tilde{D}}$ of $\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^{*}$. Since $X$ and $\tilde{Y}$ are smooth, the terms of this complex are $p$-torsion free and $p$-adically complete. The endomorphism $\phi_D$ is $\mathcal{O}_{\tilde{D}}$-linear and $\phi_{\tilde{D}}$ vanishes on $\Omega_{Y/k}^1$, hence $\phi_D^1$ is divisible by $p$ and $\phi_D^i = p^i F$ for a unique $\mathcal{O}_D$-linear endomorphism $F$ of $\mathcal{O}_D \otimes \Omega_{Y/W}^{i}$. Thus $(\mathcal{O}_D \otimes \Omega_{Y/W}^{*}, \text{d}, F)$ is a Dieudonné complex.

1It is hard to find an explicit construction of this isomorphism in [2], although of course it does follow from the comparison [2, 4.4.12] with the classical de Rham Witt complex and Illusie’s theorem [II 1.4][3]. In fact Illusie explains a new version of the proof in his note [4]. The method presented here different. I would like to thank Illusie for very helpful conversations concerning it.
**Theorem 1** Let $X/k$ be a smooth scheme over a perfect field $k$, embedded as a locally closed subscheme of a smooth $\tilde{Y}/W$ which is endowed with a lifting $\phi_{\tilde{Y}}$ of the Frobenius endomorphism of its reduction $Y/k$ modulo $p$. Then the Dieudonné complex $(O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^\cdot, d, F)$ constructed above is in fact a (torsion-free) Dieudonné algebra. Moreover, the natural map

$$(O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^\cdot, d) \to \mathcal{W}Sat(O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^\cdot, d)$$

is a quasi-isomorphism, and the natural map

$$\mathcal{W}Sat(O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^\cdot, d, F) \to (W\Omega_X^\cdot, d, F)$$

is an isomorphism. Thus, $(W\Omega_X^\cdot, d)$ is a representative of $Ru_{X/W^*}(O_{X/W})$.

**Proof:** To see that $(O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^\cdot, D, F)$ is a Dieudonné algebra, we must show that $\phi_{\tilde{D}}: \tilde{D} \to \tilde{D}$ reduces to the Frobenius endomorphism $F_D$ of $D$ [2, 3.1.2]. The reduction $\phi_D$ of $\phi_{\tilde{D}}$ is the unique PD morphism $D \to D$ extending $F_Y$, and so it will suffice to show that $F_D$ is in fact a PD-morphism. But if $t$ is an element of the PD-ideal $T$ of $X$ in $D$, then $F_D^*(t) = t^p = p!(t^p) = 0$, and hence for any $n \geq 1$, $F_D \circ \gamma_n = \gamma_n \circ F_D = 0$.

Note that $O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^\cdot$ is not the same as the the de Rham complex of $\tilde{D}$; the latter has a lot of $p$-torsion.

**Lemma 2** In the following diagram, the lower triangle commutes, even though the upper one does not. (NB: here we always mean the $p$-adically completed de Rham complexes; and in particular we are dividing by the $p$-adic closure of the torsion in the lower right hand corner.)

\[
\begin{array}{ccc}
O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^1 & \xrightarrow{t} & \Omega_{\tilde{D}/W}^1 \\
\downarrow \nabla & & \downarrow \pi \\
O_{\tilde{D}} & \xrightarrow{d} & \Omega_{\tilde{D}/W}^1/(\text{torsion})^-
\end{array}
\]

Furthermore, the composite

$$(\tilde{t}: O_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}^1 \to \Omega_{\tilde{D}/W}^1/(\text{torsion})^-)$$

is an isomorphism.
Proof: The top horizontal arrow in the diagram is induced by adjunction. The algebra \( \mathcal{O}_D \) is locally generated over \( \mathcal{O}_Y \) by the divided powers \( f^{[n]} \) of elements \( f \) of the ideal of \( X \) in \( \tilde{Y} \), for \( n \geq 1 \). For any such \( f \), we have \( \nabla f^{[n]} = f^{[n-1]} \otimes df \) in \( \mathcal{O}_D \otimes \Omega^1_{\tilde{Y}/W} \) [1, 6.4]. On the other hand, since \( n!f^{[n]} = f^n \) and \( n!f^{[n-1]} = nf^{n-1} \), we have

\[
n!df^{[n]} = d(n!f^{[n]}) = df^n = nf^{n-1}df = n(n-1)!f^{[n-1]}df = n!f^{[n-1]}df
\]

in \( \Omega^1_{D/W} \). Thus \( \nabla f^{[n]} \) and \( df^{[n]} \) have the same image in \( \Omega^1_{D/W}/(\text{torsion}) \), so the lower triangle commutes.

Since \( d: \mathcal{O}_D \to \Omega^1_{D/W} \) is the universal derivation to a \( p \)-adically complete sheaf of \( \mathcal{O}_D \)-modules, there is a unique map \( s: \Omega^1_{D/W} \to \mathcal{O}_D \otimes \Omega^1_Y \) such that \( s \circ d = \nabla \); this map factors through a map

\[
\tilde{s}: \Omega^1_{D/W}/(\text{torsion})^\sim \to \mathcal{O}_D \otimes \Omega^1_Y.
\]

Then

\[
\tilde{t} \circ s \circ d = \pi \circ t \circ s \circ d = \pi \circ t \circ \nabla = \pi \circ d,
\]

and it follows that \( \tilde{t} \circ s = \pi \) and hence that \( \tilde{t} \circ \tilde{s} = \text{id} \). On the other hand, if \( f \) is a local section of \( \mathcal{O}_Y \), then \( f \) can also be viewed as a section of \( \mathcal{O}_D \), and \( \nabla f = 1 \otimes df \) in \( \mathcal{O}_D \otimes \Omega^1_{Y/W} \). Thus the upper right triangle of the diagram does commute when restricted to \( \mathcal{O}_Y \), and it follows that \( \tilde{s} \circ \tilde{t} = \text{id} \). \( \square \)

Since \( (\tilde{D}, \phi_{\tilde{D}}) \) is a \( p \)-torsion free lifting of \( (D, F_D) \) so by [2, 3.2.1], there is an endomorphism \( F \) of the graded abelian sheaf \( \Omega^1_{D/W} \) which gives it the structure of a Dieudonné algebra.

**Lemma 3** The map \( t \) in Lemma 2 induces an isomorphism of Dieudonné algebras:

\[
w: \text{WSat}(\mathcal{O}_D \otimes \Omega^1_{Y/W}, d, F) \to \text{WSat}(\Omega^1_{D/w}, d, F).
\]

**Proof:** By construction, the natural map

\[
\text{Sat}(\Omega^1_{D/W}, d, F) \to \text{Sat}(\Omega^1_{D/W}, d, F)/(\text{torsion}).
\]

is an isomorphism, and hence so is the natural map

\[
\text{WSat}(\Omega^1_{D/W}, d, F) \to \text{WSat}(\Omega^1_{D/W}, d, F)/(\text{torsion})^\sim.
\]

since both are \( p \)-adically complete. Thus the result follows from the second statement of Lemma 2. \( \square \)
Since \((\tilde{D}, \phi_{\tilde{D}})\) is a \(p\)-torsion free lifting of \((D, F_D)\), [2, 4.2.3] implies that there is an isomorphism of Dieudonné algebras:

\[
\mathcal{W}Sat(\Omega_{\tilde{D}/W}', d, F) \rightarrow (\mathcal{W}\Omega_{\tilde{D}}, d, F).
\]

Thus we find a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W} & \xrightarrow{\bar{t}} & \Omega_{\tilde{D}/W}/(\text{torsion})^- \\
\downarrow s & & \downarrow \cong \\
\mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}) & w & \mathcal{W}Sat(\Omega_{\tilde{D}/W}') \\
& & \mathcal{W}\Omega_{\tilde{D}} \\
& g & \\
& \mathcal{W}\Omega_{X} &
\end{array}
\]

We have seen that \(\bar{t}\) and \(w\) are isomorphisms. Since \(X\) is the reduced subscheme of \(D\), the following lemma, which is a consequence of [2, 6.5.2] and also of the easier [2, 36.1], implies that \(g\) is also an isomorphism.

**Lemma 4** If \(Z\) is scheme over \(\mathbf{F}_p\), the natural map \(Z_{\text{red}} \rightarrow Z\) induces an isomorphism \(\mathcal{W}\Omega_{Z} \rightarrow \mathcal{W}\Omega_{Z_{\text{red}}}\). \(\square\)

We conclude that the natural map \(\mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}') \rightarrow \mathcal{W}\Omega_{X}\) is an isomorphism, as asserted in the second statement of Theorem 1.

It remains to prove that the map \(\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W} \rightarrow \mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}')\) is a quasi-isomorphism. The complex \(\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}'\) is not of Cartier type, and I could not find a direct reference in [2] which proves this. But it suffices to copy some of its arguments. By [1, 8.20], applied to the constant gauge \(\epsilon = 0\), the morphism \(\phi_{\tilde{D}}\) factors through a quasi-isomorphism

\[
\alpha: \mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W} \rightarrow \eta^p_{\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}}.
\]

It follows that \(\eta^p_n(\alpha)\) is a quasi-isomorphism for every \(n\), and hence that the map

\[
\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W} \rightarrow \varprojlim \eta^p_n(\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W}) = \text{Sat}(\mathcal{O}_{\tilde{D}} \otimes \Omega_{\tilde{Y}/W})
\]
is also a quasi-isomorphism. Since both complexes are $p$-torsion free, this map remains a quasi-isomorphism when reduced modulo $p^n$ for every $n$, and by [2, 2.8.1], the map

$$Sat(\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W}) \otimes \mathbb{Z}/p^n \mathbb{Z} \to WSat(\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W}) \otimes \mathbb{Z}/p^n \mathbb{Z}$$

is also a quasi-isomorphism for every $n$. We conclude that the map

$$\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W} \to WSat(\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W})$$

is a quasi-isomorphism when reduced modulo $p$. Since both sides are $p$-adically complete and $p$-torsion free, it follows that it too is a quasi-isomorphism. \qed

References


