Systems of Linear Differential Equations

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Notation

We work over an open interval:
$I := (a, b) := \{t \in \mathbf{R} : a < t < b\}$.
Fix a positive integer n, and consider the vector space of functions:

$$V := \{\mathbf{x} \colon I \to \mathbf{R}^n\}.$$

Thus an element of V has the form:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{pmatrix}.$$

Let:

- $C^0_n(I)$ be the set of of those $\mathbf{x}$ such that each x_i is continuous,
- $C^1_n(I)$ be the set of those $\mathbf{x}$ such that each derivative x_i' exists and is continuous.

These are linear subspaces of V.

Normal form for linear system of differential equations

Let

- $\mathbf{A}$ be an $n \times n$ matrix of continuous functions on I.
- $\mathbf{y}$ be an $n \times 1$ matrix of continuous functions on I, that is, an element of $C_n^0(I)$.

We consider a system of the form

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + y_2 \\
&\dots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + y_n
\end{aligned}
$$

We can write this very simply using matrix notation:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{y}$$

equivalently

$$\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{y}.$$

The existence and uniqueness theorem

Theorem

Given $\mathbf{A}$ *and* $\mathbf{y}$ *as above, then for any* $t_0 \in I$ *and any* $\mathbf{x}_0 \in \mathbf{R}^n$*, there is a unique* $\mathbf{x} \in C_n^1(I)$ *such that*

- $\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{y}$*, and*
- $\mathbf{x}(t_0) = \mathbf{x}_0$.

Why does this make sense?
Think of a walk in the park, with signposts everywhere.