Systems of Linear Differential Equations

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Notation

We work over an open interval:
$I := (a, b) := \{t \in \mathbf{R} : a < t < b\}$.
Fix a positive integer n, and consider the vector space of functions:

$$V := \{\mathbf{x} \colon I \to \mathbf{R}^n\}.$$

Thus an element of V has the form:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{pmatrix}.$$

Let:

- $C_n^0(I)$ be the set of of those $\mathbf{x}$ such that each x_i is continuous,
- $C_n^1(I)$ be the set of those $\mathbf{x}$ such that each derivative x_i' exists and is continuous.

These are linear subspaces of V.

Normal form for linear system of differential equations

Let

- $\mathbf{A}$ be an $n \times n$ matrix of continuous functions on I.
- $\mathbf{y}$ be an $n \times 1$ matrix of continuous functions on I, that is, an element of $C_n^0(I)$.

We consider a system of the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + y_2 \\ & \ldots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + y_n \end{aligned}$$

We can write this very simply using matrix notation:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{y}$$

equivalently

$$\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{y}.$$

The existence and uniqueness theorem

Theorem

Given $\mathbf{A}$ *and* $\mathbf{y}$ *as above, then for any* $t_0 \in I$ *and any* $\mathbf{x}_0 \in \mathbf{R}^n$*, there is a unique* $\mathbf{x} \in C_n^1(I)$ *such that*

- $\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{y}$*, and*
- $\mathbf{x}(t_0) = \mathbf{x}_0$.

Why does this make sense?
Think of a walk in the park, with signposts everywhere.

Linear algebra point of view

Give **A** as above, consider the mapping

$$L\colon C^1_n(I) \to C^0_n(I) \quad \mathbf{x} \mapsto \mathbf{x}' - \mathbf{A}\mathbf{x}.$$

That is, $L(\mathbf{x}) := \mathbf{x}' - \mathbf{A}\mathbf{x}.$ Then L is a *linear transformation.* Furthermore, for each $t_0 \in I$, the *evaluation mapping*

$$E_{t_0}\colon C^1_n(I) \to \mathbf{R}^n$$

is also a linear transformation.

Restatement

Theorem

- *The linear transformation*

$$L : C_n^1(I) \to C_n^0(I)$$

is surjective.

- *For any $t_0 \in I$, the map*

$$E_{t_0} : Ker(L) \to \mathbf{R}^n$$

is an isomorphism.

Corollary

The dimension of the linear subspace $Ker(L) \subseteq C_n^1(I)$ is a linear subspace of dimension n.

Corollary

Let $(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n)$ *be any linearly independent sequence in* $Ker(L)$*. Then*

- $(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n)$ *is a basis for* $Ker(L)$*.*
- *Any* $\mathbf{x} \in Ker(L)$ *can be written uniquely*

  $$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n,$$

  for some $c_i \in \mathbf{R}$*.*
- *If* $\mathbf{x}_0$ *is any solution to* $L(\mathbf{x}) = \mathbf{y}$*, then every solution* $\mathbf{x}$ *can be written uniquely*

  $$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n + \mathbf{x}_0,$$

  for some $c_i \in \mathbf{R}$*.*

The Wronskian

If $(\mathbf{x}_1, \ldots \mathbf{x}_n)$ is a sequence of elements of $Ker(L)$, let

$$W := \det \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

Corollary

Then the following conditions are equivalent:

- *For some $t_0 \in I$, $W(t_0) \neq 0$.*
- *For some $t_0 \in I$, the sequence of vectors*

$$E_{t_0}(\mathbf{x}_1), E_{t_0}(\mathbf{x}_2), \ldots, E_{t_0}(\mathbf{x}_n)$$

  in $\mathbf{R}^n$ is linearly independent.
- *$(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a basis for $Ker(L)$.*
- *$W(t) \neq 0$ for all $t \in I$.*