Matrix multiplication and composition of linear transformations

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Let $B \in M_{nq}$ and let $A \in M_{pm}$ be matrices. Note that q is the number of columns of B and is also the length of the rows of B, and that p is the number of rows of A and is also the length of the columns of A.

Definition 1 *If $B \in M_{nq}$ and $A \in M_{pm}$, the matrix product BA is defined if $q = p$. In this case it is the element of M_{nm} whose ijth entry is given by*

$$(BA)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ip}B_{pj}.$$

Thus the matrix product is an operation:

$$M_{np} \times M_{pm} \to M_{nm}.$$

Formulas:

1. If $m = 1$, multiplication by B is a map $\mathbf{R}^p = M_{p1} \to \mathbf{R}^n = M_{n1}$. It sends a column vector $X = \begin{pmatrix} x_1 \\ \cdots \\ x_p \end{pmatrix}$ to

   $$BX = x_1C_1(B) + x_2C_2(B) + \cdots x_pC_p(B),$$

   where $C_j(B)$ is the jth column of B.

2. If $m = 1$, $Be_j = C_j(B)$, where e_j is the jth element of the standard frame for $\mathbf{R}^p$.

Theorem 2 *Let $T : \mathbf{R}^p \to \mathbf{R}^n$ be any function. Then the following are equivalent:*

1. *There exists an $n \times p$ matrix B such that $T = T_B$, i.e., such that $T(X) = T_B(X)$ for all $X \in \mathbf{R}^p$.*
2. *T satisfies the principle(s) of superposition:*
   (a) *$T(X + X') = T(X) + T(X')$ for all X and X' in $\mathbf{R}^p$, and*
   (b) *$T(cX) = cT(X)$ for all $X \in \mathbf{R}^p$ and $c \in \mathbf{R}$.*

Proof: The proof that (1) implies (2) was given earlier and is in the book; it is a consequence of the properties of matrix multiplication. Let us prove that (2) implies (1). First note that if (2) is true and if $X_1, \ldots, X_r$ are vectors and $c_1, \ldots c_r$ are numbers, then $T(c_1X_1 + \cdots + c_rX_r) = c_1T(X_1) + \cdots + c_rT(x_r)$. Now if T satisfies (2), we let B the $n \times p$ matrix whose jth column is the vector $T(e_j)$, for $j = 1, \ldots p$. We must prove that $T(X) = T_B(X)$ for all $X \in \mathbf{R}^p$. But if $X \in \mathbf{R}^p$, $X = x_1e_1 + \cdots x_pe_p$, where the x_i's are the coordinates of X. Hence

$$\begin{aligned} T(X) &= T(x_1e_1 + \cdots x_pe_p) \\ &= x_1T(e_1) + \cdots x_pT(e_p) \\ &= x_1C_1(B) + \cdots x_pC_p(B) \\ &= BX \\ &= T_B(X) \end{aligned}$$

□

Corollary 3 *A linear transformation is uniquely determined by its effect on the standard frame. More precisely, if T and T' are linear transformations from $\mathbf{R}^p$ to $\mathbf{R}^n$ and if $T(e_j) = T'(e_j)$ for all j, then $T(X) = T'(X)$ for all X.*

Corollary 4 *The composite of two linear transformations is linear.*

Proof: Suppose S and T are linear and composible. Then

$$\begin{aligned} S \circ T(X + X') &= S(T(X + X')) \\ &= S(T(X) + T(X')) \\ &= S(T(X)) + S(T(X')) \\ &= S \circ T(X) + S \circ T(X'). \end{aligned}$$

The proof of the compatibility with scalar multiplication is similar. □

Theorem 5 *If $B \in M_{np}$ and $A \in M_{pm}$, then the composite $T_B \circ T_A$ is T_{BA}.*

Proof: We know that $T_B \circ T_A$ and T_{BA} are linear. To prove that they are equal it suffices to check that they have the same effect on each e_j. We compute

$$\begin{aligned} T_B \circ T_A(e_j) &= T_B(T_A(e_j)) \\ &= T_B(C_j(A)) \\ &= BC_j(A) \\ &= C_j(BA) \\ &= T_{BA}(e_j) \end{aligned}$$

□

Corollary 6 *Matrix multiplication is associative. That is if C, B and A are matrices with the correct dimensions, then $(CB)A = C(BA)$.*

Theorem 7 *If A and B are $n \times n$ matrices such that $BA = I_n$ (the identity matrix), then B and A are invertible, and $B = A^{-1}$.*

Proof: Suppose that $BA = I_n$. Let us prove that the rank of A is n. To do this it suffices to check that T_A is injective. But $T_B \circ T_A = T_{BA} = T_{I_n}$ is the identity transformation. Thus if $T_A(X) = T_A(X')$, then $X = T_B(T_A(X)) = T_B(T_A(X'))$, so $X = X'$, *i.e.*, T_A is injective, A has rank n, and hence A is invertible. Let A^{-1} be its inverse. We have $I_n = BA$, so

$$A^{-1} = I_n A^{-1} = (BA)A^{-1} = B(AA^{-1}) = BI_n = B.$$

□