The inverse of a matrix

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Theorem 1 *Let A be an $n \times n$ matrix of rank n. Then T_A is invertible, and its inverse is $T_{A^{-1}}$, where A^{-1} can be computed as follows. Let I_n be the $n \times n$ identity matrix and let $(A \quad I_n)$ be the $n \times 2n$ matrix formed by adding the columns of I_n to the right of A_n. Then $rref(A \quad I_n) = (I_n \quad A^{-1})$.*

Proof: We have already proved that T_A is invertible. Recall also that since A has rank n, all the rows of $rref(A)$ are nonzero, hence every row and every column has a leading index, hence $rref(A) = I_n$. Let A^{-1} be the matrix described above. To show that $T_{A^{-1}}$ is the inverse of T_A, it is enough to show that for any $Y \in \mathbf{R}^n$, $X := T_{A^{-1}}Y$ satisfies the equation $T_A X = Y$. Let us first check this when $Y = e_j$, where $(e_1, \dots e_n)$ be the standard frame for $\mathbf{R}^n$. To solve the equation $T_A(X) = e_j$, one puts the augmented matrix $(A \quad e_j)$ in reduced row echelon form. This will look like $(I_n \quad C_j)$, where C_j is some column vector, and in fact C_j is the solution: $T_A(C_j) = e_j$. Now you can see easily that all the C_j's can be calculated together by using the method of the theorem: C_j is just the jth column of the matrix A^{-1} described above.

To deduce the general case we use the principle of superposition. For any Y, $Y = \sum_j y_j e_j$, where the y_j's are the entries of Y. Hence the principle of superposition tells us that

$$T_{A^{-1}}(Y) = \sum_j y_j T_{A^{-1}}(e_j).$$

Recall that for any matrix B, $T_B(e_j) = C_j(B)$, the jth column of B. Thus

$$T_{A^{-1}}(Y) = \sum_j y_j C_j(A^{-1}).$$

Now by the principle of superposition again,

$$T_A(T_{A^{-1}}(Y)) = T_A(\sum_j y_j C_j(A^{-1})) = \sum_j y_j T_A(C_j A^{-1}).$$

By what we saw above, $T_A(C_j A^{-1}) = e_j$, so

$$T_A(T_{A^{-1}}(Y)) = \sum_j y_j T_A(C_j(A^{-1})) = \sum_j y_j e_j = Y$$

□