Matrix Algebra and Linear Equations

August 31, 2007

To solve a system $\mathcal{E}$ of n linear equations in m unknowns using Gauss elimination:

- Write the $n \times (m+1)$ augmented matrix $\tilde{A}$ which represents $\mathcal{E}$.
- Perform a series of elementary row operations to transform $\tilde{A}$ into an augmented matrix $\tilde{A}'$ in reduced row echelon form.
- Analyze the system $\mathcal{E}'$ corresponding to $\tilde{A}'$.

This method implicitly uses the following important fact (theorem). We will not repeat its proof here.

Theorem 1 *Let $\tilde{A}$ be a matrix.*

- *There is sequence of elementary row operations which transforms $\tilde{A}$ into a matrix $\tilde{A}'$ in reduced row echelon form.*
- *The systems of equations $\mathcal{E}$ and $\mathcal{E}'$ corresponding to $\tilde{A}$ and $\tilde{A}'$ are equivalent, i.e, they have the same solution set.*

Remark: In fact there is a well-defined *algorithm* that a computer (or you) can use which tells you exactly which sequence of row operations to use to get from a matrix A to a matrix in reduced row echelon form. Thus we can write $rref(A)$, meaning that $rref(A)$ is a well-defined *function* of A. But in fact more is true; *any* sequence of elementary row operation which yields a matrix in reduced row echelon form when applied to A will give the same answer. More precisely:

Theorem 2 *Let A, A', and A'' be matrices. Assume that A' and A'' are in reduced row echelon form and that both A' and A'' can be obtained from A by a sequence of elementary row operations. Then $A' = A''$.*

The proof of this theorem is not in the book. I may have time to explain it later.

The rank of a matrix and the number of solutions of a system

Definition 3 *The rank of a matrix is the number of nonzero rows in its reduced row echelon form.*

Proposition 4 *Let A be an $n \times m$ matrix. Then*

$$\begin{aligned} 0 &\leq rank(A) \leq n \\ 0 &\leq rank(A) \leq m \end{aligned}$$

Proof: The reduced row echelon form A' of A also has n rows, so $r \leq n$. Since the leading entries of the nonzero rows of A' occur in different columns, there must also be at least r different columns in A'. Since A' has m columns, $r \leq m$. □

How many solutions does a system have? We expect that the more equations there are, the fewer the solutions there are. But we should only count "independent equations." One of our long term goals is to make this notion precise. In the meantime we can say the following.

Let $\tilde{A}$ be the augmented $n \times (m + 1)$ matrix corresponding to a system of n equations in m unknowns. Let $\tilde{A}' := rref(\tilde{A})$ and let A' be the first m columns of $\tilde{A}'$; in fact $A' = rref(A)$, where A is the first m columns of $\tilde{A}$. Let r be the number of nonzero rows of A and let $\tilde{r}$ be the number of nonzero rows of $\tilde{A}$. These look something like:

$$\begin{pmatrix} 1 & * & * & 0 & * & | & 0 \\ 0 & 0 & 0 & 1 & * & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & * & * & 0 & * & | & 0 \\ 0 & 0 & 0 & 1 & * & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Theorem 5 *Let $\tilde{A}$ be the augmented $n \times (m+1)$ matrix corresponding to a system $\mathcal{E}$ of n equations and m unknowns, and let A be the $n \times m$ coefficient matrix consisting of the first m columns of $\tilde{A}$.*

- *The system $\mathcal{E}$ is consistent (has at least one solution) if and only if $\tilde{A}$ and A have the same rank.*
- *If this is true and $r := rank(\tilde{A}) = rank(A)$, then there are $m - r$ free variables and r fixed variables.*

Proof: The system $\mathcal{E}$ is inconsistent if and only the system $\mathcal{E}'$ is inconsistent, and this is true if and only if it contains an equation of the form $0 = 1$. This is true if and only if $\tilde{A}'$ has a row with a leading entry in the last column. Thus $\mathcal{E}$ is *consisten* if and only if $\tilde{A}'$ does *not* have such a row. This means exactly that all the leading entries of all the nonzero rows of $\tilde{A}'$ in fact lie in A', that is, if and only if $\tilde{A}'$ and A' have the same number of nonzero rows. By the definition of rank, this is true if and only if $\tilde{A}$ and A have the same rank.

Now suppose this is the case. Then there are r nonzero rows, each one of which allows us to solve for one of the leading index variables (the "fixed variables"), and the remaining $m - r$ variables are "free." □

Corollary 6 *Any system $\mathcal{E}$ has either no solutions, exactly one solution, or infinitely many solutions.*

□

Corollary 7 *Let $\mathcal{E}$ be a system of n equations in m unknowns. Let $\tilde{A}$ be the augmented matrix corresponding to $\mathcal{E}$ and let A be the coefficient matrix.*

- *If $n < m$, $\mathcal{E}$ has either no solutions or infinitely many solutions.*
- *If $m = n$ and is equal to the rank of the coefficient matrix corresponding to $\mathcal{E}$, then $\mathcal{E}$ has exactly 1 solution.*

Note that in the last case, the numbers appearing on the right sight of the equations (which will appear in the last column of the augmented matrix expressing the equations), do not affect the number of solutions.

Proof: If $\mathcal{E}$ is consistent, then the ranks of $\tilde{A}$ and A are the same number r, which is less than or equal to n and hence striclty less than m. Thus the number $m - r$ of free variables is positive, so there are infinitely many solutions. This proves the first statement. For the second statement, suppose that the rank r of A is equal to n and is also equal to m. Then since n is the number of rows of $\tilde{A}$,

$$n = rank(A) \leq rank(\tilde{A}) \leq n.$$

This implies that the equations are consistent. Moreover, the number of free variables is $m - r = 0$, so there is exactly one solution. □