Orthogonal matrices

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The following formula is extremely useful. It left as an exercise in our book, but I think it is too important for that.

Theorem 1 *Let* $v \in \mathbf{R}^m$, $w \in \mathbf{R}^n$ *and* $A \in M_{nm}(\mathbf{R})$. *Then*

$$(A^T v|w) = (v|Aw).$$

Proof:

$$\begin{aligned} (A^T v|w) &= (A^T v)^T w \\ &= (v^T A^{T^T})w \\ &= (v^T A)w \\ &= v^T (Aw) \\ &= (v|Aw). \end{aligned}$$

□

Theorem 2 *Let* A *be an* $n \times n$ *matrix. Then the following conditions are equivalent:*

1. *The columns* $(v_1, \dots v_n)$ *of* A *form an orthonormal sequence.*
2. $A^T A = I_n$ *(the* $n \times n$ *identity matrix).*
3. $(Av|Aw) = (v|w)$ *for every* $v, w \in \mathbf{R}^n$.

Proof: Suppose (1) holds. Let us compute the ijth entry of $B := A^T A$. This is obtained by multiplying the transpose of the ith row of A^T by the jth column of A. That is

$$\begin{aligned} b_{ij} &= R_i(A^T)C_j(A) \\ &= C_i(A)^T C_j(A) \\ &= v_i^T v_j = (v_i|v_j) \end{aligned}$$

Since $(v_1, \ldots v_n)$ is orthonormal, this is 1 if $i = j$ and is zero otherwise. In other words, $B = I_n$, and (2) holds. Suppose (2) holds. Then for any v, w,

$$(Av|Aw) = (A^T Av|w) = (I_n v|w) = (v|w).$$

Suppose (3) holds. Recall that $v_i = Ae_i$, where $(e_1, \ldots e_n)$ is the standard frome for $\mathbf{R}^n$. Then

$$(v_i|v_j) = (Ae_i|Ae_j) = (e_i|e_j),$$

which is 1 if $i = j$ and is zero otherwise. Thus $(v_1, \ldots v_n)$ is orthonormal. $\square$