Inner products

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Inner products are used to deal with two important geometric notions: angle and distance.

Definition 1 *Let v and w be vectors in $\mathbf{R}^n$, with coordinates $(x_1, \dots x_n)$ and $(y_1, \dots y_n)$, respectively. Then the inner product or dot product of v and w is:*

$$v \cdot w := \langle v, w \rangle := (v|w) := x_1 y_1 + x_2 y_2 + \cdots x_n y_n.$$

Note that there are at least three sets of notation commonly used to denote the inner product. Furthermore, it is sometimes called the "dot product" or "scalar product." Note that the dot product of two vectors in $\mathbf{R}^n$ is a real number, and that the matrix product $v^T w$ is the 1×1 matrix whose only entry is $(v|w)$. We shall sometimes identify the matrix $v^T w$ with the number $(v|w)$.

Theorem 2 *The inner product $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ satisfies the following properties:*

1. $(v + v'|w) = (v|w) + (v'|w)$ *if* $v, v', w \in \mathbf{R}^n$.
2. $(av|w) = a(v|w)$ *if* $v, w \in \mathbf{R}^n$ *and* $a \in \mathbf{R}$.
3. $(v|w) = (w|v)$ *if* $v, w \in \mathbf{R}^n$.
4. $(v|v) > 0$ *if* $v \neq 0$, *for any* $v \in \mathbf{R}^n$ *and* $(0|0) = 0$.

These properties are quite easy to verify. Note however that the last of them uses the special fact that the square of any real number is positive.

Definition 3 *If $v \in \mathbf{R}^n$, then $||v|| := \sqrt{(v|v)}$. The (nonnegative) real number $||v||$ is called the magnitude or length of v. If v and w are two elements $\mathbf{R}^n$, then $||v-w||$ is called the distance between v and w. If $v, w \in \mathbf{R}^n$, then $v \perp w$ if $(v|w) = 0$, in which case we say that v and w are orthogonal.*

Proposition 4 *If $v, w \in \mathbf{R}^n$, then*

1. $||v+w||^2 = ||v||^2 + ||w||^2 + 2(v|w)$.
2. *If $v \perp w$, then $||v+w||^2 = ||v||^2 + ||w||^2$.*
3. $|(v|w)| \leq ||v||\,||w||$.
4. $||v+w|| \leq ||v|| + ||w||$.

Of these, the only difficult one is (3), the Cauchy-Schwartz inequality. It is easy to prove (4) from (3) and (1).

Definition 5 *A sequence of vectors $(v_1, \ldots v_m)$ in $\mathbf{R}^n$ is orthogonal if $v_i \perp v_j$ whenever $i \neq j$. The sequence is orthonormal if it is orthogonal and in addition $||v_i|| = 1$ for all i.*

The following result is probably the most important theorem about inner products.

Theorem 6 *Let W be a linear subspace of $\mathbf{R}^n$ and let v be a member of $\mathbf{R}^n$. There v can be written uniquely*

$$v = \pi_W(v) + \pi_W^{\perp}(v),$$

where $\pi_W(v) \in W$ and $\pi_W^{\perp}(v)$ is orthogonal to every vector in W. Furthermore:

1. *$\pi_W(v)$ is the vector in W which is closest to v. That is,*

   $$||\pi_W(v) - v|| \leq ||w - v|| \qquad \text{for every } w \in W,$$

   with equality only if $w = \pi_W(v)$.

2. *If $(w_1, \ldots w_m)$ is an orthonormal basis for W, then*

   $$\pi_W(v) = (v|w_1)w_1 + \cdots (v|w_m)w_m.$$

3. More generally, if $(w_1, \dots w_m)$ if an orthogonal basis for W, then

$$\pi_W(v) = \frac{(v|w_1)w_1}{||w_1||^2} + \cdots \frac{(v|w_m)w_m}{||w_m||^2}.$$

Let me explain a proof of (1). Let $w_0 := \pi_W(v)$ This a vector in W. Let w be any other vector in W. We claim that if $w \neq w_0$, then $||v - w|| > ||v - w_0||$. Let $w' := w_0 - w$. This is another vector in W, and

$$\begin{aligned} ||v - w||^2 &= ||(v - w_0) + (w_0 - w)||^2 \\ &= ||\pi^\perp(v) + w'||^2 \\ &= ||\pi^\perp(v)||^2 + ||w'||^2 \end{aligned}$$

since $\pi^\perp(v)$ is orthogonal to w' (use formula). But $||w'||^2 \geq 0$, and is zero only if $w = w_0$.