Coordinates

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Let V be a linear subspace of $\mathbf{R}^n$ and let $\beta := (v_1, \ldots v_m)$ be a basis for V. Then every vector v in V can be written uniquely as a linear combination:

$$v = c_1 v_1 + \cdots c_m v_m.$$

The numbers c_i are called the *coordinates of v with respect to the basis β.* It is convenient to arrange these into a column, and write

$$[v]_\beta := \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_m \end{pmatrix}.$$

Then the vector $[v]_\beta$ is called the *coordinate vector of v with respect to the basis β.*

Theorem: The mapping $v \mapsto [v]_\beta$ defines a bijection between the linear subsapce V and $\mathbf{R}^m$. Furthermore, this bijection is compatible with vector addition and scalar multiplication:

$$[v + v']_\beta = [v]_\beta + [v']_\beta$$

$$[cv]_\beta = c[v]_\beta.$$

Now let V be a linear subspace of some $\mathbf{R}^M$ and W a linear subspace of some $\mathbf{R}^N$, and suppose further that we have bases $\beta := (v_1, \ldots v_m)$ for V and $\gamma := (w_1, \ldots w_n)$ for W. Then if $T \colon V \to W$ is a linear transformation, we define the *matrix of T with respect to the bases β and γ*. This is the $n \times m$ matrix whose jth column is the coordinate vector $[T((v_j)]_\gamma$ of $T(v_j)$ with respect to the basis γ. This matrix is often denoted by $[T]_\beta^\gamma$, although some authors put β above and γ below. If $V = W$, it is usually best to take $\beta = \gamma$ as well.

The most important formula to remember is the following.

Let $\beta := (v_1, \dots v_n)$ be a basis for $\mathbf{R}^n$. Let S be the $n \times n$ matrix whose columns are the v_j's. Then S is invertible. Moreover, if T is the linear transformation T_A where A is some $n \times n$ matrix, and B is the matrix for T with respect to the basis β, then

$$SB = AS, \quad B = S^{-1}AS, \quad \text{and } A = SBS^{-1}$$

Fortunately you need only remember the first of these, as the remaining two are easy to deduce. To figure out why the first is true, just compute column by column. The equation says $C_j(SB) = C_j(AS)$, that is $SC_j(B) = AC_j(S)$. On the right side, we have $AC_j(S) = T(v_j)$. On the left, if $C_j(B)$ is the column vector whose coordinates are $(x_1, \dots x_n)$ we have $SC_j(B) = x_1 v_1 + \cdots x_n v_n$. This means exactly the the x_i's are the coordinates of $T(V_j)$ in the basis β, as in the definition.