The rank nullity theorem

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Let A be an $n \times m$ matrix. We would like to use Gauss elimination to find a basis for the column space of A. In order to do this we need to understand more about the relationship between A and its reduced row echelon form.

Definition 1 *Two matrices A and A' are said to be row equivalent if there exists a sequence of elementary row operations which transforms A to A'.*

Note that A is row equivalent to itself. Furthermore, if A is row equivalent to A', A' is row equivalent to A, because elementary row operations are reversible. Finally, note that if A is row equivalent to A' and A' is row equivalent to A'', then A is row equivalent to A'', since we can chain together the necessary sequences of elementary row operations.

Proposition 2 *Let A and A' be $n \times m$ matrices. Then A and A' are row equivalent if and only if there exists an invertible $n \times n$ matrix U such that $A' = UA$.*

Let me just explain how to find U. (In fact there can be many such U.) This is similar to the algorithm we used for inverting a matrix, only here we do not require that A be square. Just form the $n \times (m + n)$ matrix $(A|I_n)$, and apply the elementary row operations necessary to change A to A'. The matrix you get is $(A'|U)$, and $UA = A'$. This is easy to see if A' can be obtained by a single elementary row operation just by checking, and the general case follows by chaining.

Corollary 3 *If A' and A are row equivalent, then*

1. *A' and A have the same null space.*

2. *A' and A have the same row space (the span of the rows).*

Proof: Actually we have already verified the first fact. As an exercise, deduce it from proposition 2. The second fact is easily seen if A' is obtained from A by a single elementary row operation just from the definitions, and the general case follows by a chaining argument. □

It is now possible to prove that the reduced row echelon form of a matrix is unique. Look for the note on this on the web page. Our main goal here is to show how to use Gauss elimination to find a basis for the column space of a matrix. The difficulty is that the column space (unlike the row space) changes when elementary row operations are performed.

Theorem 4 *Let A be an $n \times m$ matrix and let A' be its reduced row echelon form.*

1. *The null space of A is equal to the null space of A'.*
2. *The rank r of A is equal to the rank of A'.*
3. *If $(\ell_1, \dots \ell_r)$ are the leading indices of A', then the corresponding columns $(C_{\ell_1}(A'), \dots C_{\ell_r}(A'))$ forms a basis for the column space of A', and*
4. *$(C_{\ell_1}(A), \dots C_{\ell_r}(A))$ forms a basis for the column space of A.*

The third statement is easy to see by inspection of the reduced row echelon matrix A. The last statement is not explained well in the book. It is easy to understand it, using the proposition, since $A = UA'$ for some invertible matrix U. For any j, $C_j(A')$ can be written as a linear combination of the leading index columns of A'. Then

$$\begin{aligned}
C_j(A') &= x_1 C_{\ell_1}(A') + \cdots x_r C_{\ell_r}(A') \\
UC_j(A') &= U\left(x_1 C_{\ell_1}(A') + \cdots x_r C_{\ell_r}(A')\right) \\
&= x_1 UC_{\ell_1}(A') + \cdots x_r UC_{\ell_r}(A') \\
C_j(UA') &= x_1 C_{\ell_1}(UA') + \cdots x_r C_{\ell_r}(UA') \\
C_j(A) &= x_1 C_{\ell_1}(A) + \cdots x_r C_{\ell_r}(A)
\end{aligned}$$

This shows that every column of A is a linear combination of the leading index columns, so they span the column space. A similar argument, using U^{-1} shows that there are no redundancies among these columns, so they form a basis for the column space of A.