Spans, Linear Independence and Bases

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Definition 1 *Let* $(v.) := (v_1, \ldots, v_m)$ *be a sequence of vectors in* $\mathbf{R}^n$*. Then a vector* $v \in \mathbf{R}^n$ *is said to be a linear combination of* $v.$ *if there exists a sequence of numbers* $(c.) := (c_1, \ldots, c_m)$ *such that*

$$v = c_1 v_1 + c_2 v_2 + \cdots c_m v_m.$$

The set of all such v *is called the span of* $(v.)$*.*

Note that if A is the $n \times m$ matrix whose columns are the vectors v_i, then v is in the span of $(v.)$ if and only if there exists a column vector X in $\mathbf{R}^m$ such that $v = AX$, *i.e.*, if and only if v is in the image of T_A. Thus the span of $(v.)$ is always a linear subspace of $\mathbf{R}^n$.

The span of the empty sequence is defined to be the set consisting of just the zero vector.

Definition 2 *Let* $(v.) := (v_1, \ldots, v_m)$ *be a sequence of vectors. Then for* $1 \leq i \leq m$*, the* i*th vector* v_i *is said to be redundant in* $(v.)$ *if it belongs to the span of the sequence* $(v_1, \ldots, v_{i-1})$*. A sequence* $(v.)$ *is said to be linearly dependent if it contains a redundant vector; otherwise it is said to be linearly independent.*

Lemma 3 *If* $(v_1, \ldots v_m)$ *is a sequence of vectors, then for each* i*, the following are equivalent.*

1. v_i *is redundant.*

2. $span(v_1, \ldots, v_{i-1}) = span(v_1, \ldots, v_i)$*.*

It follows from the lemma that we can omit all the redundant vectors from a sequence $(v.)$ without changing the span. It is pretty clear that the sequence we obtain in this way has no redundant vectors, *i.e.*, is linearly independent.

Proposition 4 *Let $(v.) := (v_1, \ldots, v_m)$ be a sequence of vectors. Then the following conditions are equivalent:*

1. *$(v.)$ is linearly independent.*
2. *If $(c.) := (c_1, \ldots c_m)$ is a sequence of numbers such that $c_1v_1 + \cdots v_mv_m = 0$, each $c_i = 0$.*
3. *If $(c.)$ and $(c'.)$ are two sequences of numbers such that $c_1v_1 + \cdots c_mv_m = c'_1 + \cdots c'_mv_m = 0$, then $c_i = c'_i$ for all i.*

This can be translated into matrix terms for computational purposes. Arrange the vectors v_i into the columns of an $n \times m$ matrix A. Consider the column vector X whose coordinates are the c_i appearing in the proposition above and recall that

$$AX = x_1C_1(A) + x_2C_2(A) + \cdots x_mC_m(A) = c_1v_1 + \cdots c_mv_m.$$

Then condition (2) says that the kernel of T_A is $\{0\}$, and condition (3) says that T_A is injective.

Corollary 5 *Let A be an $n \times m$ matrix. Then the following conditions are equivalent.*

1. *The columns of A are linearly independent.*
2. *The kernel of T_A (the nullspace of A) is $\{0\}$.*
3. *T_A is injective.*
4. *The rank of A is m.*

Recall that the rank of a matrix is always less than or equal to the the length of its columns.

Corollary 6 *If $v_1, \ldots v_m$ is an independent sequence of vectors in $\mathbf{R}^n$, then $m \leq n$.*

Definition 7 *Let W be a linear subspace of $\mathbf{R}^n$. A sequence $(w_\cdot)$ of vectors in W is a* basis *for W if it is linearly independent and spans W.*

Thus if $(w_\cdot)$ is a basis for W, ever vector w in W can be expressed *uniquely* as a linear combination $c_1 w_1 + \cdots + c_m w_m$.